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Multivariate Vertex Splines and Finite Elements*

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The objective of this paper is to present a unified study of multivariate super vertex splines with emphasis on the construction procedure and an application to least-squares approximation with interpolatory constraints. Both simplicial and parallelepiped partitions are studied in some detail, and in the bivariate setting, even a partition consisting of both triangles and parallelograms is considered. When the polynomial degree is allowed to be sufficiently large as compared to the order of smoothness, it is clear that vertex splines can be constructed by working on each simplex or parallelepiped separately as long as certain suitable normal derivative constraints are imposed on the boundary faces. Our constructive procedure will take a different route. Instead of normal derivatives, we impose extra interpolatory conditions at the "vertices." This gives rise to the notion of "super splines" introduced in this paper. It should also be emphasized that the view point of considering a basis of piecewise polynomials with smallest possible supports so that the full approximation order is preserved makes vertex splines different from the standard approach in finite elements. After all, if the polynomial degree is required to be lower, it is necessary to work on at least three adjacent simplices or parallelepipeds simultaneously in constructing a basis of vertex splines.

1. INTRODUCTION

It is well known that (polynomial) spline functions in one variable provide an extremely useful tool in any theoretical or applied research and computational endeavor that requires any form of approximation of only partially or even implicitly known functions of one variable. Extensive studies on both the theory and its applications are avaialable in the vast spline literature (cf. [25, 4, 27]). Recently, there has also been considerable progress in the study of multivariate spline functions (or more precisely,

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piecewise polynomial functions satisfying certain smoothness conditions) (cf. [9]). In particular, box splines provide a natural and computationally efficient generalization of univariate B-splines on equally spaced knot. To generalize univariate B-splines on an arbitrary knot sequence to the multivariate setting, so that important problems such as treatment of scattered data can be handled, a natural approach is to give a basis of compactly supported piecewise polynomial functions on a given simplicial partition. This problem, however, is extremely complicated, and a general approach does not seem to be feasible. For this reason, the notion of bivariate *vertex splines* was introduced in [11] in order to give a generalization of the univariate C^1 cubic and C^2 quintic Hermite basis to the two-dimensional setting, one advantage being that vertex splines are easily computable. The objectives of this paper are to present a unified study of vertex splines in any number of dimensions, including both simplicial and parallelepiped partitions (and in the two-dimensional setting, even mixed partitions), and to discuss an application to least-squares approximation with interpolatory constraints. For completeness, even some known results (usually in different versions) will be included and verified in this paper, although appropriate references will also be provided. Most of the results in this paper have been announced in [12]. When the degree d of the polynomial pieces in s-variables is much larger than the order r of smoothness, such as $d \geqslant 2^s r + 1$ as already suggested by [29, 30, 21], the construction of vertex splines will be seen to be intimately related to the methods in finite elements. Hence, the notion of super splines is introduced. These are C^r piecewise polynomial functions with higher order of smoothness across lower-dimensional manifolds of the grid of partition. It will be seen that at least for $d \geqslant 2^s r + 1$, the subspace of super splines already gives the full order of approximation, namely $d+1$. It should be noted and emphasized that the notion of vertex splines is not confined to the restriction of $d \geqslant 2^s r + 1$. Indeed, it is the point of view of considering a basis of smooth piecewise polynomials with smallest possible supports that separates the study of vertex splines from the standard procedure in working on each simplex or parallelepiped individually. For instance, in [13], when bivariate piecewise polynomials of total degree d on an arbitrary triangulation are considered, the collection of all vertex splines in C^r cannot be obtained by using the standard procedure of the finite element method when $d = 3r + 2$ and $r \geqslant 2$. For this reason, vertex splines will provide an important vehicle to introduce spline techniques to the methods of finite elements. However, our study of lower degree vertex splines must be delayed to a later date (cf. [13] for $s = 2$). It should also be noted that vertex splines are constructed only when a grid partition is already given. Many methods for generating simplicial partitions can be found in the literature (cf. [28]).

The outline of this paper is as follows. Bézier and Bernstein representations of polynomials on simplices and parallelepipeds will first be discussed. An approach to the use of interpolation conditions at the vertices to determine a polynomial on a simplex or parallelepiped will be introduced in Section 3. Section 4 will be devoted to the study of smoothness conditions of piecewise polynomials. Here, known results in perhaps different formulations are included for both completeness and convenience. The main section is Section 5, where vertex splines are defined, construction procedures are given for the case $d \geqslant 2^s r + 1$, and that full approximation order is achieved by super spline subspaces via vertex splines is verified. In Section 6, least-squares approximation with interpolatory constraints will be studied. Examples and graphs on various supports are shown in the last section.

2. Polynomial Representations

Let $\mathbf{Z}_+^s$ denote the set of all multi-integers with non-negative components in the Euclidean space $\mathbf{R}^s$, where $s \geqslant 1$. As usual, for $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbf{Z}_+^s$, we will use the notations $|\alpha| = \alpha_1 + \cdots + \alpha_s$, $\alpha! = \alpha_1! \cdots \alpha_s!$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ for any $\mathbf{x} = (x_1, ..., x_s) \in \mathbf{R}^s$. In addition, for another $\beta = (\beta_1, ..., \beta_s) \in \mathbf{Z}_+^s$, $\beta \leqslant \alpha$ will mean $\beta_i \leqslant \alpha_i$ for all $i = 1, ..., s$.

We will not follow the usual way,

$$P(\mathbf{x}) = \sum_{\substack{\alpha \in \mathbf{Z}_+^s \\ \text{finite numbers of } \alpha \neq 0}} a_\alpha \mathbf{x}^\alpha,$$

to express a polynomial $P(\mathbf{x})$, but instead we will use the Bézier polynomial representation on a somplex and the Bernstein polynomial representation on a parallelepiped. Such representations are independent of the Cartesian coordinates and hence provide more convenient expressions for our study of piecewise polynomials. This section is divided into two parts so that we can study each representation in some detail.

2.1. *The Simplex Case*

Let $\mathbf{x}^0, ..., \mathbf{x}^s \in \mathbf{R}^s$, $s \geqslant 1$. The convex hull

$$T_1 = \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle = \left\{ \sum_{i=0}^{s} \lambda_i \mathbf{x}^i : \sum_{i=0}^{s} \lambda_i = 1, \lambda_i \geqslant 0 \right\}$$

of the set $\{\mathbf{x}^0, ..., \mathbf{x}^s\}$ is called an s-simplex if its (directed) s-dimensional volume

$$\operatorname{vol}_s \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle = \frac{1}{s!} \begin{vmatrix} 1 & x_1^0 & \cdots & x_s^0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^s & \cdots & x_s^s \end{vmatrix}$$

is nonzero. Here and throughout, we set $\mathbf{x}^i = (x_1^i, ..., x_s^i)$, $i = 0, ..., s$. Suppose that $\langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle$ is an s-simplex. Then any $\mathbf{x} = (x_1, ..., x_s)$ in $\mathbf{R}^s$ can be identified by an $(s+1)$-tuple $(\lambda_0, ..., \lambda_s)$, where

$$\lambda_i = \lambda_i(\mathbf{x}) = \frac{\mathrm{vol}_s \langle \mathbf{x}^0, ..., \mathbf{x}^{i-1}, \mathbf{x}, \mathbf{x}^{i+1}, ..., \mathbf{x}^s \rangle}{\mathrm{vol}_s \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle}.$$

This $(s+1)$-tuple is called the barycentric coordinate of $\mathbf{x}$ relative to the s-simplex T_1.

Note that each $\lambda_i = \lambda_i(\mathbf{x})$ is a linear polynomial in $\mathbf{x}$. Hence, for $\beta \in \mathbf{Z}_+^{s+1}$ with $|\beta| = n$, where $n \in \mathbf{Z}_+$,

$$\phi_\beta^n(\lambda) = \frac{n!}{\beta!} \lambda^\beta$$

is a polynomial in $\pi_n^s(T_1)$, the space of all polynomials in s-variables of total degree $\leqslant n$ with respect to T_1. In fact, it is easy to see that $\{\phi_\beta^n(\lambda): |\beta| = n\}$ is a basis of $\pi_n^s(T_1)$. The polynomial

$$P_n(\mathbf{x}) = \sum_{|\beta| = n} a_\beta^n \phi_\beta^n(\lambda) \tag{2.1.1}$$

is called a *Bézier polynomial* of total degree n relative to the s-simplex T_1. In addition, the set

$$\left\{ \left(\sum_{i=0}^{s} \frac{\beta_i}{n} \mathbf{x}^i, a_\beta^n \right) : |\beta| = n \right\}, \tag{2.1.2}$$

and for brevity $\{a_\beta^n\}$, is called the *Bézier net* of the polynomial P_n. Hence, to describe the polynomial P_n, we simply write down its Bézier net on the simplicial array. For example, in Figure 2.1.1, we show the Bézier net of a polynomial in $\pi_4^2(T_1)$ on a triangular array.

Let us first consider the properties of differentiation and integration of Bézier polynomials. If f is a differentiable function, and A and B are two

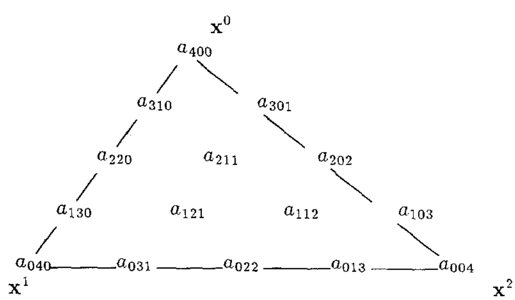

FIG. 2.1.1. The Bézier net of P_4 in $\mathbf{R}^2$

distinct points in $\mathbf{R}^s$, the derivative of f along the directed line segment from A to B at $\mathbf{x}$ is denoted by

$$(D_{B-A}f)(\mathbf{x}) = \frac{d}{dt} f(\mathbf{x} + t(B-A))|_{t=0}$$

$$= (B-A) \cdot \left(\frac{\partial}{\partial x_1} f(\mathbf{x}), ..., \frac{\partial}{\partial x_s} f(\mathbf{x}) \right).$$

Hence, if $\mathbf{y} = (y_1, ..., y_s) = B - A$, we have

$$D_{\mathbf{y}} = \sum_{i=1}^{s} y_i \frac{\partial}{\partial x_i}.$$

If $\mathbf{y} = \mathbf{x}^i - \mathbf{x}^j$, where $\mathbf{x}^i \neq \mathbf{x}^j$, however, we will also use the notation

$$D_{ij} = D_{\mathbf{x}^i - \mathbf{x}^j}, \qquad i \neq j.$$

To discuss differences, we will use the notation

$$E_i a_\alpha = a_{(\alpha_0, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_s)},$$

where the $(i+1)$st component of the index $\alpha = (\alpha_0, ..., \alpha_s)$ is advanced by 1, and we introduce the difference operator

$$\Delta_{ij} a^n_\alpha = E_i a^n_\alpha - E_j a^n_\alpha.$$

We have

LEMMA 2.1.1. *For $i \neq j$,*

$$(D_{ij} P_n)(\mathbf{x}) = n \sum_{|\alpha| = n-1} \Delta_{ij} a^n_\alpha \phi^{n-1}_\alpha(\lambda). \tag{2.1.3}$$

Proof. To prove this lemma, we recall that if $\mathbf{x}^i = (x^i_1, ..., x^i_s)$ and $\mathbf{x} = (x_1, ..., x_s)$, then

$$x_l = \sum_{t=0}^{s} \lambda_t x^t_l, \qquad l = 1, ..., s,$$

so that

$$(D_{ij} P_n)(\mathbf{x}) = \sum_{l=1}^{s} (x^i_l - x^j_l) \frac{\partial}{\partial x_l} P_n(\mathbf{x})$$

$$= \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j} \right) P_n(\mathbf{x}).$$

Hence, (2.1.3) follows from a simple change of indices in

$$\sum_{\alpha_0+\cdots+\alpha_s=n} \frac{n!}{\alpha_0!\cdots\alpha_s!} a_\alpha^n \left(\frac{\partial}{\partial\lambda_i}-\frac{\partial}{\partial\lambda_j}\right)(\lambda_0^{\alpha_0}\cdots\lambda_s^{\alpha_s}).$$

For integration of a polynomial on an s-simplex, we have the following result.

LEMMA 2.1.2. *For any $\beta\in\mathbf{Z}_+^{s+1}$ with $|\beta|=n$,*

$$\int_{\langle \mathbf{x}^0,\ldots,\mathbf{x}^s\rangle} \phi_\beta^n(\lambda_0(\mathbf{x}),\ldots,\lambda_s(\mathbf{x}))\,d\mathbf{x}=\frac{|\mathrm{vol}_s\langle \mathbf{x}^0,\ldots,\mathbf{x}^s\rangle|}{\binom{n+s}{s}}. \tag{2.1.4}$$

Proof. Equation (2.1.4) follows immediately from a change of variables and an integral formula of the multi-Γ function.

Consequently, we have

COROLLARY 2.1.1. *Let*

$$P_n(\mathbf{x})=\sum_{\substack{|\beta|=n\\ \beta\in\mathbf{Z}_+^{s+1}}} b_\beta\phi_\beta^n(\lambda_0(\mathbf{x}),\ldots,\lambda_s(\mathbf{x})).$$

Then

$$\int_{\langle \mathbf{x}^0,\ldots,\mathbf{x}^s\rangle} P(\mathbf{x})\,d\mathbf{x}=\frac{|\mathrm{vol}_s\langle \mathbf{x}^0,\ldots,\mathbf{x}^s\rangle|}{\binom{n+s}{s}}\sum_{\substack{|\beta|=n\\ \beta\in\mathbf{Z}_+^{s+1}}} b_\beta. \tag{2.1.5}$$

Also, observing that

$$\phi_\beta^n(\lambda_0,\ldots,\lambda_s)\,\phi_\alpha^m(\lambda_0,\ldots,\lambda_s)=\frac{\binom{\beta+\alpha}{\beta}}{\binom{m+n}{n}}\phi_{\beta+\alpha}^{m+n}(\lambda_0,\ldots,\lambda_s),$$

we have the following formula for the inner product of two polynomials over an s-simplex.

COROLLARY 2.1.2. *Let*

$$P_n(\mathbf{x})=\sum_{\substack{|\beta|=n\\ \beta\in\mathbf{Z}_+^{s+1}}} b_\beta\phi_\beta^n(\lambda_0(\mathbf{x}),\ldots,\lambda_s(\mathbf{x}))$$

and

$$Q_m(\mathbf{x})=\sum_{\substack{|\beta|=m\\ \beta\in\mathbf{Z}_+^{s+1}}} c_\alpha\phi_\alpha^m(\lambda_0(\mathbf{x}),\ldots,\lambda_s(\mathbf{x})).$$

Then

$$\int_{\langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle} P_n(\mathbf{x})\, Q_m(\mathbf{x})\, d\mathbf{x} = \frac{|\operatorname{vol}_s \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle|}{\binom{m+n+s}{s}\binom{m+n}{n}} \sum_{\substack{|\beta| = n \\ |\alpha| = m}} b_\beta c_\alpha \binom{\beta+\alpha}{\beta}. \qquad (2.1.6)$$

We refer to [5] and [16] for some other properties of Bézier polynomials. To evaluate a polynomial in Bézier representation, we may apply the de Casteljau algorithm (see, e.g., [7, 2, 15, 3, 14]). However, to graphically display a Bézier polynomial surface P_n, we may use the Bézier nets on subdivisions of T_1 instead of the exact values of P_n on T_1. Efficient algorithms are available and will be discussed elsewhere (see, e.g., [9].)

2.2. *The Parallelepiped Case*

Let $\{\mathbf{x}^1, ..., \mathbf{x}^{2^s}\}$ be a set of 2^s distinct points in $\mathbf{R}^s$ so chosen that its convex hull $T_2 = \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$ is a parallelepiped with s-dimensional volume $\operatorname{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle \neq 0$, and call T_2 an s-parallelepiped. In this subsection, we consider only non-negative scalar-valued s-dimensional volumes. Clearly, the $(s-1)$-dimensional boundary of the s-parallelepiped T_2 consist of $2s$ $(s-1)$-parallelepipeds, $A_1, ..., A_{2s}$, say. Suppose that they are so ordered that $A_{2k-1} \parallel A_{2k}$ (i.e., A_{2k-1} is parallel to A_{2k}), $k = 1, ..., s$. For $\mathbf{x} \in \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$, we let $\operatorname{vol}_s \langle A_k, \mathbf{x} \rangle$ be the s-dimensional volume of the convex hull of $\{\mathbf{x}, A_k\}$, $k = 1, ..., 2s$. Then we have

$$\frac{\operatorname{vol}_s \langle A_{2k-1}, \mathbf{x} \rangle}{\operatorname{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle} + \frac{\operatorname{vol}_s \langle A_{2k}, \mathbf{x} \rangle}{\operatorname{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle} = \frac{1}{s},$$

$k = 1, ..., s$. Set

$$v_k = v_k(\mathbf{x}) = s \frac{\operatorname{vol}_s \langle A_{2k-1}, \mathbf{x} \rangle}{\operatorname{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle}, \qquad k = 1, ..., s.$$

Then the barycentric coordinate of $\mathbf{x}$ relative to $T_2 = \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$ is $(v_1, ..., v_s)$. Thus, we may consider polynomials $\tilde{P}_\alpha(\mathbf{x})$ of coordinate degree $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbf{Z}_+^s$ in the form of

$$\tilde{P}_\alpha(\mathbf{x}) = \sum_{\gamma \leqslant \alpha} \tilde{a}_\gamma^\alpha \tilde{\phi}_\gamma^\alpha(v), \qquad (2.2.1)$$

where

$$\tilde{\phi}_\gamma^\alpha(v) = \binom{\alpha}{\gamma} v^\gamma (1-v)^{\alpha-\gamma} \qquad (2.2.2)$$

with

$$\binom{\alpha}{\gamma} := \binom{\alpha_1}{\gamma_1} \cdots \binom{\alpha_s}{\gamma_s},$$

$\gamma = (\gamma_1, ..., \gamma_s)$.

If $\alpha=(n, \ldots, n)$, $n \in \mathbf{Z}_{+}$, we will simply write

$$\tilde{P}_n=\tilde{P}_{(n, \ldots, n)}, \qquad \tilde{\phi}_\gamma^n=\tilde{\phi}_\gamma^{(n, \ldots, n)}, \qquad \tilde{a}_\gamma^n=\tilde{a}_\gamma^{(n, \ldots, n)}.$$

Also, let $\tilde{\pi}_\alpha^s(T_2)$ denote the space of all such polynomials $\tilde{P}_\alpha$ and $\tilde{\pi}_n(T_2)=\tilde{\pi}_{(n, \ldots, n)}(T_2)$. For convenience, let us also assume that $\mathbf{x}^1 \in \bigcap_k A_{2k-1}$ and $\mathbf{x}^{i+1} \in A_{2i}$, $i=1, \ldots, s$, such that $v_j(\mathbf{x}^{i+1})=\delta_{ij}$, $i, j=1, \ldots, s$. Then the polynomial $\tilde{P}_\alpha(\mathbf{x})$ in (2.2.1) is called a *Bernstein polynomial* of coordinate degree α relative to the parallelepiped $T_2=\langle \mathbf{x}^1, \ldots, \mathbf{x}^{2^s}\rangle$, and the set

$$\left\{\mathbf{x}^0+\sum_{\substack{\gamma \leqslant \alpha \\ i=1, \ldots, s}} \frac{\gamma_i}{\alpha_i}(\mathbf{x}^{i+1}-\mathbf{x}^1) : \tilde{\alpha}_\gamma^\alpha\right\}, \tag{2.2.3}$$

or for brevity $\{\tilde{\alpha}_\gamma^\alpha\}$, is called the *Bézier net* of $\tilde{P}_\alpha$ relative to this parallelepiped T_2. In Fig. 2.2.1, we represent a polynomial in $\tilde{\pi}_4^2(T_2)$ in terms of its Bézier net on a parallelogram array.

We now introduce some properties on differentiation and integration of the Bernstein polynomials $\tilde{P}_\alpha$. We have two lemmas which follow immediately from the corresponding univariate results. The notation

$$\tilde{D}_i=D_{\mathbf{x}^{i+1}-\mathbf{x}^i}$$

will be used. In addition, we will set $\Delta\tilde{a}_\gamma=\tilde{a}_{\gamma+\mathbf{e}^i}-\tilde{a}_\gamma$ where $\mathbf{e}^i=(0, \ldots, 0, 1, 0, \ldots, 0)$ is the standard unit vector in $\mathbf{R}^s$ with 1 in the ith component.

LEMMA 2.2.1. *Let* $\tilde{P}_\alpha$ *be given as in* (2.2.1). *Then*

$$\tilde{D}_i \tilde{P}_\alpha(\mathbf{x})=\alpha_i \sum_{\gamma \leqslant \alpha-e^i} \Delta_i \tilde{a}_\gamma \tilde{\phi}_\gamma^{\alpha-e^i}(v_1(\mathbf{x}), \ldots, v_s(\mathbf{x})). \tag{2.2.4}$$

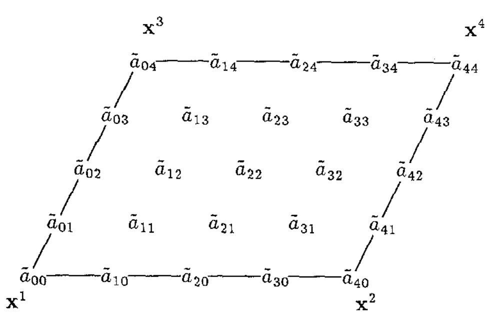

FIG. 2.2.1. The Bézier net of $\tilde{P}_4$ in $\mathbf{R}^2$

LEMMA 2.2.2. *For each* $\gamma \leqslant \alpha$,

$$\int_{\langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle} \tilde{\phi}_\gamma^\alpha(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))\, d\mathbf{x} = \frac{\text{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle}{(\alpha_1 + 1) \cdots (\alpha_s + 1)}. \tag{2.2.5}$$

Hence, for $\tilde{P}_\alpha(x) = \sum_{\gamma \leqslant \alpha} \tilde{a}_\gamma \tilde{\phi}_\gamma^\alpha(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$ on a parallelepiped $\langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$, we have

COROLLARY 2.2.1.

$$\int_{\langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle} \tilde{P}_\alpha(\mathbf{x})\, d_{\mathbf{x}} = \frac{\text{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle}{(\alpha_1 + 1) \cdots (\alpha_s + 1)} \sum_{\gamma \leqslant \alpha} \tilde{a}_\gamma, \tag{2.2.6}$$

Observing that

$$\tilde{\phi}_\gamma^\alpha(v_1, ..., v_s)\, \tilde{\phi}_\delta^\beta(v_1, ..., v_s) = \frac{\binom{\gamma+\delta}{\gamma}}{\binom{\alpha+\beta}{\alpha}} \tilde{\phi}_{\gamma+\delta}^{\alpha+\beta}(v_1, ..., v_s),$$

we also have

COROLLARY 2.2.2. *For any two polynomials*

$$\tilde{P}_\alpha(\mathbf{x}) = \sum_{\gamma \leqslant \alpha} a_\gamma \tilde{\phi}_\gamma^\alpha(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$$

and

$$\tilde{Q}_\beta(x) = \sum_{\delta \leqslant \beta} c_\delta \tilde{\phi}_\delta^\beta(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$$

on $T_2 = \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$, *we have*

$$\int_{\langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle} \tilde{P}_\alpha(\mathbf{x})\, \tilde{Q}_\beta(\mathbf{x})\, d\mathbf{x}$$
$$= \frac{\text{vol}_s \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle}{(\alpha_1 + \beta_1 + 1) \cdots (\alpha_s + \beta_s + 1)} \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} \frac{\binom{\gamma+\delta}{\gamma}}{\binom{\alpha+\beta}{\alpha}} a_\gamma c_\delta. \tag{2.2.7}$$

To evaluate the value of $\tilde{P}_\alpha(\mathbf{x})$ at some $\mathbf{x} \in \langle \mathbf{x}^1, ..., \mathbf{x}^{2^s} \rangle$, we may use de Casteljau's algorithm a number of times (cf. [3]).

3. POLYNOMIAL INTERPOLATION

In this section, we will develop a theory of multivariate interpolation by Bézier and Bernstein polynomials. The results in this section will be used to facilitate our procedure in constructing multivariate vertex splines. The interpolation will be taken at vertices of a simplex or a parallelepiped, and we will express the interpolation polynomials in terms of the Bézier nets. Since we will use polynomials of both total degree and coordinate degree, we have to treat them separately and employ different notations. For polynomials of total degree, we consider interpolation at vertices of an s-simplex, and for polynomials of coordinate degree, we consider interpolation at vertices on an s-parallelepiped.

Throughout this section, we will use the following definition: a subset $M^s \in \mathbf{Z}_+^s$ is called a lower set if $\gamma \in M^s$ whenever $\beta \in M^s$ and $0 \leqslant \gamma \leqslant \beta$. The following theorem gives an inversion formula which will be frequently used in this and the next section.

THEOREM 3.1. *Let M^s be a lower set in $\mathbf{Z}_+^s$ and suppose that*

$$f(\alpha) = \sum_{0 \leqslant \gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-1)^{|\gamma|} g(\gamma), \qquad \alpha \in M^s.$$

Then

$$g(\alpha) = \sum_{0 \leqslant \gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-1)^{|\gamma|} f(\gamma), \qquad \alpha \in M^s.$$

3.1. *The Simplex Case*

In this subsection, we will always assume that $T_1 = \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle$ is an s-simplex. We need the following additional notation: for $\beta \in \mathbf{Z}_+^s$, let

$$D_0^\beta := D_{10}^{\beta_1} \cdots D_{s0}^{\beta_s},$$

$$D_i^\beta := D_{0i}^{\beta_1} \cdots D_{i-1,i}^{\beta_i} D_{i+1,i}^{\beta_{i+1}} \cdots D_{si}^{\beta_s} \qquad i = 1, ..., s,$$

and

$$D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_s}\right)^{\beta_s}.$$

Also, for $\alpha \in \mathbf{Z}_+^{s+1}$, let c_i be a map from $\mathbf{Z}_+^{s+1}$ to $\mathbf{Z}_+^s$ defined by

$$c_i \alpha = c_i(\alpha_0, ..., \alpha_s) = (\alpha_0, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_s),$$

where $i \in \{0, ..., s\}$.

We are ready to state and prove the following theorem.

THEOREM 3.1.1. *In Bézier representation with respect to $\langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle$, the Taylor polynomial of a sufficiently smooth function f at the vertex $\mathbf{x}^0$ is given by*

$$P_n(f, \mathbf{x}) = \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbf{Z}_+^{s+1}}} \sum_{\substack{\beta \leqslant c_0 \alpha \\ \beta \in \mathbf{Z}_+^s}} \binom{c_0 \alpha}{\beta} \frac{(n - |\beta|)!}{n!} D_0^\beta f(\mathbf{x}^0)\, \phi_\alpha^n(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x})). \quad (3.1.1)$$

Proof. Let

$$P_n(f, \mathbf{x}) = \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbf{Z}_+^{s+1}}} a_\alpha^n \phi_\alpha^n(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x}))$$

be the Taylor polynomial of function f at $\mathbf{x}^0$. Then for each $\beta \in \mathbf{Z}_+^s$ with $|\beta| \leqslant n$,

$$D_0^\beta P_n(f, \mathbf{x}^0) = D_0^\beta f(\mathbf{x}^0).$$

By Lemma 2.1.1, we see that

$$\begin{aligned}(-1)^{|\beta|} P_n(f, \mathbf{x}^0) &= \frac{n!}{(n-|\beta|)!} (-1)^{|\beta|} \Delta_{10}^{\beta_1} \cdots \Delta_{s0}^{\beta_s} a_{(n-|\beta|, 0, ..., 0)} \\ &= \frac{n!}{(n-|\beta|)!} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} a_{(n-|\gamma|, \gamma_1, ..., \gamma_s)}.\end{aligned}$$

Hence, applying the inversion formula in Theorem 3.1, we obtain

$$\begin{aligned}a_{(n-|\beta|, \beta_1, ..., \beta_s)} &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} \frac{(n-|\gamma|)!}{n!} (-1)^{|\gamma|} D_0^\gamma P_n(f, x^0) \\ &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} D_0^\gamma f(x^0),\end{aligned}$$

completing the proof of the theorem.

In general, we also have the folowing formulation of the interpolation polynomial at each vertex of the simplex.

THEOREM 3.1.2. *Suppose that all partial derivatives up to order k_i of a function f at $\mathbf{x}^i$ exist. Let*

$$\begin{aligned}p_{n,k_i}(\mathbf{x}) := \sum_{\substack{|\alpha|=k_i \\ \alpha \in \mathbf{Z}_+^{s+1}}} \sum_{\substack{\beta \leqslant c_i\alpha \\ \beta \in \mathbf{Z}_+^s}} \binom{c_i\alpha}{\beta} \frac{(n-|\beta|)!}{n!} \\ \times D_i^\beta f(\mathbf{x}^i)\, \phi_{\alpha+(n-k_i)e^i}^n(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x}))\end{aligned}$$

for $i = 0, ..., s$. Then the polynomial

$$p_n(f, \mathbf{x}) = \sum_{i=0}^s p_{n,k_i}(\mathbf{x}) \tag{3.1.2}$$

in $\pi_n^s(T_1)$ satisfies the interpolation conditions

$$D_i^\beta p_n(f, \mathbf{x}^i) = D_i^\beta f(\mathbf{x}^i), \qquad |\beta| \leqslant k_i, \tag{3.1.3}$$

for $i = 0, ..., s$ and $\beta \in \mathbf{Z}_+^s$, provided $n \geqslant \max\{k_i + k_j, i \neq j\} + 1$.

Proof. It is obvious that we need only verify that p_{n,k_i} satisfies that

$$D_i^\beta p_{n,k_i}(\mathbf{x}^i) = D_i^\beta f(\mathbf{x}^i), \qquad |\alpha| \leqslant k_i \tag{1}$$

and

$$D_j^\beta p_{n,k_i}(\mathbf{x}^j) = 0, \qquad |\alpha| \leqslant k_j, j \neq i. \tag{2}$$

Clearly, (1) can be verified by the inversion formula in Theorem 3.1 along the lines of the proof of Theorem 3.1.1. To prove (2), we note that for $n > k_i + k_j$ and $|\beta| \leqslant k_j$,

$$D_j^\beta \phi^n_{(\alpha_0, ..., \alpha_{i-1}, \alpha_i + n - k_i, \alpha_{i+1}, ..., \alpha_s)}(\lambda_0(\mathbf{x}^j), ..., \lambda_s(\mathbf{x}^j)) = 0$$

for $\alpha \in \mathbf{Z}_+^{s+1}$ with $|\alpha| \leqslant k_i$. That is, (2) holds and we have established the theorem.

In the following, let $N_{ij} \in \mathbf{Z}_+$ and $M_i^s = \{\beta \in \mathbf{Z}_+^s : \beta_j \leqslant N_{ij}, j = 1, ..., s\}$, $i = 0, ..., s$. Set $n = (s+1)N + 1$ where $N = \max\{N_{ij} : i = 0, ..., s, j = 1, ..., s\}$. Then we have

THEOREM 3.1.3. *Suppose that f is a sufficiently smooth function. Then the polynomial*

$$p_n(f, x) = \sum_{i=0}^{s} \sum_{\gamma \in M_i^s} \sum_{\beta \leqslant \gamma} \binom{\gamma}{\beta} \frac{(n - |\beta|)!}{n!} \times D_i^\beta f(\mathbf{x}^i)\, \phi^n_{(\gamma_i, ..., \gamma_i, n - |\gamma|, \gamma_{i+1}, ..., \gamma_s)}(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x})) \tag{3.1.4}$$

satisfies the interpolation condition

$$D_i^\beta p_n(f, \mathbf{x}^i) = D_i^\beta f(\mathbf{x}^i), \qquad \beta \in M_i^s, i = 0, ..., s. \tag{3.1.5}$$

Proof. Let

$$p_{n,i}(\mathbf{x}) = \sum_{\gamma \in M_i^s} \sum_{\beta \leqslant \gamma} \binom{\gamma}{\beta} \frac{(n - |\beta|)!}{n!} \times D_i^\beta f(\mathbf{x}^i)\, \phi^n_{(\gamma_1, ..., \gamma_i, n - |\gamma|, \gamma_{i+1}, ..., \gamma_s)}(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x})).$$

Then we must prove that $p_{n,i}$ satisfies

$$D_i^\beta p_{n,i}(\mathbf{x}^i) = D_i^\beta f(\mathbf{x}^i), \qquad \beta \in M_i^s \tag{3}$$

and

$$D_j^\beta p_{n,i}(\mathbf{x}^j) = 0, \qquad \beta \in M_j^s, j = 1, ..., s. \tag{4}$$

Once (3) and (4) are established, the theorem then follows.

Since $\beta \in M_j^s$ and $|\gamma| \leqslant sN < (s+1)N + 1 - \beta_i$, $\gamma \in M_i^s$, we have

$$D_j^\beta \phi^n_{(\gamma_1, \ldots, \gamma_i, n-|\gamma|, \gamma_{i+1}, \ldots, \gamma_s)}(\lambda_1(\mathbf{x}^j), \ldots, \lambda_s(\mathbf{x}^j)) = 0$$

for all $\gamma \in M_j^s$ and $j = 1, \ldots, s$. Hence, $p_{n,i}(\mathbf{x})$ satisfies (4). To verify (3), we may again apply the inversion formula as in the proof of Theorem 3.1.1.

Remark. [21] obtained a particular case of Theorem 3.1.2 and Theorem 3.1.3 generalizes a result in [17] which was used to construct blending interpolation. In general, our interpolation polynomials are not uniquely determined by the interpolation conditions (3.1.3) and (3.1.5), but in Theorems 3.1.2 and 3.1.3, we have explicit formulas of interpolation polynomials in terms of Bézier representations. Of course, for $s = 1$, Theorem 3.1.2 and Theorem 3.1.3 give the same (unique) interpolation polynomial determined by (3.1.3).

EXAMPLE 3.1.1. Let $s = 1$. The polynomial $p_n(f, x)$ satisfying

$$D^i p_n(f, 0) = D^i f(0), \qquad i = 0, \ldots, k_1$$

and

$$D^i p_n(f, 1) = D^i f(1), \qquad i = 0, \ldots, k_2,$$

where $D^i = d^i/dx^i$ with $n = k_1 + k_2 + 1$, can be written in the Bernstein representation

$$p_n(f, x) = \sum_{i=0}^{k_1} \sum_{\nu=0}^{i} \binom{i}{\nu} \frac{(n-\nu)!}{n!} f^{(\nu)}(0) \binom{n}{i} x^i (1-x)^{n-i}$$
$$+ \sum_{j=0}^{k_2} \sum_{\mu=0}^{j} \binom{j}{\mu} \frac{(n-\mu)!}{n!} (-1)^\mu f^{(\mu)}(1) \binom{n}{j} (1-x)^j x^{n-j}.$$

In the following theorems, we will specify certain interpolation conditions on the vertices of an s-simplex to ensure unique polynomial interpolation. To do so, we need some additional notation.

Let $\Gamma_n^s := \{\beta \in \mathbf{Z}_+^s : |\beta| \leqslant n\}$ and $\Lambda_n^{s+1} := \{\alpha \in \mathbf{Z}_+^{s+1} : |\alpha| = n\}$. A collection of subsets $M_0^s, \ldots, M_s^s$ of Γ_n^s is said to form a partition of Λ_n^{s+1} if the subsets satisfy:

(1) $A_i^n M_i^s \cap A_j^n M_j^s = \varnothing$ for $i \neq j$, and

(2) $\bigcup_{i=0}^s A_i^n M_i^s = \Lambda_n^{s+1}$,

where A_i^n maps $\mathbf{Z}_+^s$ to $\mathbf{Z}_+^{s+1}$ and is defined by

$$A_i^n \beta = (\beta_1, \ldots, \beta_i, n - |\beta|, \beta_{i+1}, \ldots, \beta_s), \qquad \beta \in M_i^s.$$

We have the following result.

THEOREM 3.1.4. *Suppose that $M_0^s, \ldots, M_s^s$ are lower sets that form a partition of Λ_n^{s+1}. Then for any given data $\{f_{i\beta}: \beta \in M_i^s, i=0, \ldots, s\}$, there exists a unique polynomial p_n of total degree n satisfying*

$$D_i^\beta p_n(x^i) = f_{i,\beta}, \qquad \beta \in M_i^s, i=0, \ldots, s. \tag{3.1.6}$$

Moreover, $p_n(\mathbf{x})$ may be formulated as

$$p_n(\mathbf{x}) = \sum_{i=0}^{s} \sum_{\beta \in M_i^s} \left\{ \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} f_{i\gamma} \right\} \phi_{A_i^n \beta}^n(\lambda_0(\mathbf{x}), \ldots, \lambda_s(\mathbf{x})). \tag{3.1.7}$$

Proof. Let $p_n(\mathbf{x}) = \sum_{|\alpha|=n} a_\alpha \phi_\alpha^n(\lambda_0(\mathbf{x}), \ldots, \lambda_s(\mathbf{x}))$ be a polynomial of total degree n. By Lemma 2.1.1 and the inversion formula in Theorem 3.1 as in the proof of Theorem 3.1.1, we obtain

$$a_{A_i^s \beta} = \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} f_{i\gamma}, \qquad \beta \in M_i^s.$$

Since each M_i^s is a lower set, $a_{A_i^n \beta}$' is uniquely determined by the data $\{f_{i\gamma}: \gamma \in M_i^s\}$ for all $\beta \in M_i^s$. In other words, the coefficients a_α, $\alpha \in A_i^n M_i^s$, $i=0, \ldots, s$, are uniquely determined by $\{f_{i\gamma}: \gamma \in M_i^s, i=0, \ldots, s\}$. Since $M_0^s, \ldots, M_s^s$ form a partition of Λ_n^{s+1}, the given data set $\{f_{i\gamma}: \gamma \in M_i^s, i=0, \ldots, s\}$ in (3.16) uniquely determines the interpolation polynomial.

Actually, the requirement on the sets M_i^s, $i=0, \ldots, s$, can be slightly relaxed. We have

THEOREM 3.1.5. *Suppose that $M_i^s \in \Gamma_n^s$, $i=0, \ldots, s$, form a partition of Γ_n^s. Furthermore, suppose that*

1° M_0^s is a lower set, and

2° The union of M_j^s and some subset of $c_j(\bigcup_{i=0}^{j-1} A_i^n M_i^s)$ is a lower set for $j=1, \ldots, s$. Then for any given data $\{f_{i\beta}: \beta \in M_i^s, i=0, \ldots, s\}$, there exists a unique polynomial p_n of total degree n that satisfies

$$D_i^\beta p_n(\mathbf{x}^i) = f_{i\beta} \qquad \beta \in M_i^s, i=0, \ldots, s.$$

This theorem may be proven similarly to Theorem 3.1.4 by noting that the previous information can be used in determining the remaining Bézier net of $p_n(\mathbf{x})$.

EXAMPLE 3.1.2. Let $s=2$ and $n=5$. We choose lower sets $M_0^2 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$, $M_1^2 = M_0^2$, and $M_2^2 = \{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1), (1,2), (2,1), (2,2)\}$. Then we can find a unique polynomial p_5 satifying

$$D_i^\beta p_5(\mathbf{x}^i) = f_{i\beta}, \qquad \beta \in M_i^2, i=0, 1, 2,$$

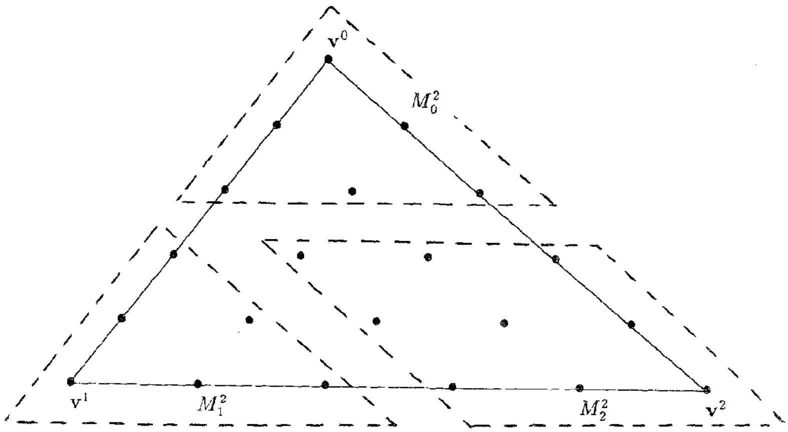

FIGURE 3.1.1

for any given data set $\{f_{i\beta}: \beta \in M_i^2, i=0, 1, 2\}$. In Figure 3.1.1, we group the Bézier net according to the corresponding M_i^2, $i=0, 1, 2$.

EXAMPLE 3.1.3. Let $s=2$ and $n=6$. We choose the sets $M_0^2=\{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (2, 1), (3, 0), (3, 1)\}$, $M_1^2=\{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3)\}$, and $M_2^2=\{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 2)\}$. By Theorem 3.1.5, we may determine the interpolation polynomial p_6 that satisfies the conditions

$$D_i^\beta p_6(\mathbf{x}^i)=f_{i\beta}, \qquad \beta \in M_i^2, i=0, 1, 2,$$

for any given data $\{f_{i\beta}: \beta \in M_i^2, i=0, 1, 2\}$. In Figure 3.1.2, we group the Bézier net according to the corresponding M_i^2, $i=0, 1, 2$.

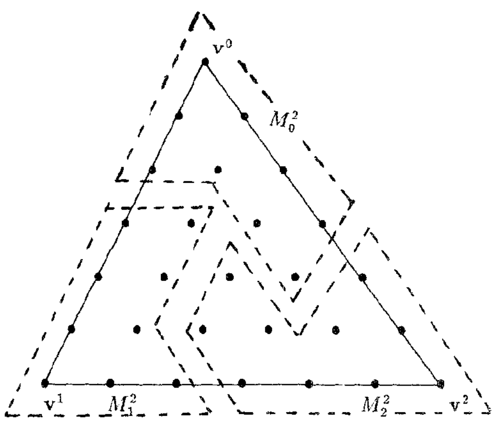

FIGURE 3.1.2

We next give an application of Theorem 3.1.1. Suppose that we have two s-simplices $S=\langle \mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^s\rangle$ and $\tilde{S}=\langle \mathbf{x}^0, \mathbf{y}^1, \ldots, \mathbf{y}^s\rangle$ sharing a common vertex $\mathbf{x}^0$ and a polynomial $p_n(\mathbf{x})=\sum_{|\alpha|=n} a_\alpha \phi_\alpha^n(\lambda_0(\mathbf{x}), \ldots, \lambda_s(\mathbf{x}))$ with respect to S. We want to find the Bézier representation of this polynomial p_n with respect to $\tilde{S}$. To do so, write $\mathbf{y}^j-\mathbf{x}^0=\sum_{i=1}^s c_{ji}(\mathbf{x}^i-\mathbf{x}^0), j=1, \ldots, s$, where

$$c_{ji}=\frac{\mathrm{vol}_s\langle \mathbf{x}^0, \ldots, \mathbf{x}^{i-1}, \mathbf{y}^j, \mathbf{x}^{i+1}, \ldots, \mathbf{x}^s\rangle}{\mathrm{vol}_s\langle \mathbf{x}^0, \ldots, \mathbf{x}^s\rangle}.$$

Let $\mathbf{c}_j=(c_{j1}, \ldots, c_{js})$, $j=1, \ldots, s$, and $\hat{D}_0^\beta=(D_{y^1-x^0})^{\beta_1}\cdots(D_{y^s-x^0})^{\beta}$, for $\beta\in \mathbf{Z}_+^s$. Then since

$$(D_{y^j-x^0}f)(\mathbf{x}^0)=\sum_{i=0}^s c_{ji}(D_{x^i-x^0}f)(\mathbf{x}^0), \qquad j=1, \ldots, s,$$

we have

$$\begin{aligned}\hat{D}_0^\beta f(\mathbf{x}^0)&=\prod_{j=1}^s\left(\sum_{i=1}^s c_{ji}D_{x^i-x^0}\right)^{\beta_j} f(\mathbf{x}^0)\\ &=\sum_{|\gamma|=|\beta|} C_\gamma^\beta D_0^\gamma f(\mathbf{x}^0) \qquad (3.1.8)\end{aligned}$$

for some constants C_γ^β. Also, since $D_0^\beta p_n(\mathbf{x}^0)=n!/(n-|\beta|)!\,\Delta_{10}^{\beta_1}\cdots\Delta_{s0}^{\beta_s} a_{(n-|\beta|,0,\ldots,0)}$, we may apply Theorem 3.1.1 to obtain

THEOREM 3.1.6. *Let $S=\langle \mathbf{x}^0, \ldots, \mathbf{x}^s\rangle$ and $\hat{S}=\langle \mathbf{y}^0, \ldots, \mathbf{y}^s\rangle$ be two simplices with a common vertex $\mathbf{x}^0=\mathbf{y}^0$. Suppose that $p_n=\sum_{|\alpha|=n} a_\alpha\phi_\alpha^n(\lambda_0, \ldots, \lambda_s)$ is a polynomial of total degree $\leqslant n$ with respect to S. Then the Bézier representation of p_n with respect to $\hat{S}$ is given by*

$$p_n(\mathbf{x})=\sum_{|\alpha|=n}\left\{\sum_{\beta\leqslant c_0\alpha}\binom{c_0\alpha}{\beta}\sum_{|\gamma|=|\beta|} C_\gamma^\beta \Delta_{10}^{\gamma_1}\cdots\Delta_{s0}^{\gamma_s} a_{(n-|\beta|,0,\ldots,0)}\right\}\phi_\alpha^n(v_0, \ldots, v_s),$$

where $\mathbf{x}=\sum_{i=0}^s v_i(\mathbf{x})\,\mathbf{y}^i$ with $\sum_{i=0}^s v_i(\mathbf{x})\equiv 1$ and the C_j^β's are defined as in (3.1.8).

3.2. *The Parallelepiped Case*

We adopt the following convention and notation in addition to those introduced in Section 2.2. Let $S=\langle \mathbf{x}^1, \ldots, \mathbf{x}^{2^s}\rangle$ be an s-parallelepiped. For each $\mathbf{x}\in S$, the barycentric coordinate of x will be denoted by

$$v=(v_1(\mathbf{x}), \ldots, v_s(\mathbf{x})),$$

where we assume that $v_i(\mathbf{x}^1)=0$, $i=1, ..., s$, and $v_i(\mathbf{x}^{i+1})=1$, $i=1, ..., s$, as before.

For each i, $1 \leqslant i \leqslant 2^s$, let $\langle \mathbf{x}^i, \mathbf{x}^{i_1}\rangle, ..., \langle \mathbf{x}^i, \mathbf{x}^{i_s}\rangle$ be the s edges of S with common vertex at $\mathbf{x}^i$ so that $\langle \mathbf{x}^i, \mathbf{x}^{i_j}\rangle \| \langle \mathbf{x}^1, \mathbf{x}^{j+1}\rangle$, $i=1, ..., s$. Hence, we may designate for each vertex $\mathbf{x}^i$ an index $\eta^i=(\eta^i_1, ..., \eta^i_s)$, where

$$\eta^i_j=\begin{cases}1 & \text{if} \quad \mathbf{x}^i-\mathbf{x}^{i_j}=\mathbf{x}^1-\mathbf{x}^{j+1};\\ -1 & \text{if} \quad \mathbf{x}^i-\mathbf{x}^{i_j}=-\mathbf{x}^1+\mathbf{x}^{j+1}.\end{cases}$$

For $\beta=(\beta_1, ..., \beta_s)\in \mathbf{Z}^s_+$, we denote by $\tilde{D}^\beta$ the differentiation operator

$$\tilde{D}^\beta=\sum_{j=0}^{s} (D_{\mathbf{x}^{j+1}-\mathbf{x}^1})^{\beta_j},$$

and for any α, $\beta \in \mathbf{Z}^s_+$ and a constant c, we use the notation

$$\alpha * \beta=(\alpha_1\beta_1, ..., \alpha_s\beta_s)\in \mathbf{Z}^s_+$$

and

$$c\beta=(c\beta_1, ..., c\beta_s)\in \mathbf{Z}^s_+.$$

Also, as before, let

$$\tilde{\pi}^s_{\mathbf{n}}(S)=\left\{\sum_{\beta \leqslant \mathbf{n}} a_\beta \tilde{\phi}^n_\beta(v_1, ..., v_s) : a_\beta \in \mathbf{R}\right\}$$

be the space of polynomials on S of coordinate degree $\mathbf{n}$, where $\mathbf{n}=(n_1, ..., n_s)\in \mathbf{Z}^s_+$. Write $\Gamma^s_{\mathbf{n}}=\{\beta \in \mathbf{Z}^s_+ : \beta \leqslant \mathbf{n}\}$ and define a one-to-one map $R^{\mathbf{n}}_i$ from $\Gamma^s_{\mathbf{n}}$ into itself by

$$R^{\mathbf{n}}_i : \alpha \mapsto \alpha * \eta^i+\frac{(\mathbf{1}-\eta^i) * \mathbf{n}}{2},$$

where $\mathbf{1}=(1, ..., 1)\in \mathbf{Z}^s_+$, and let $(R^{\mathbf{n}}_i)^{-1}$ be its inverse.

The following two theorems exhibit the collection of interpolation polynomials satisfying certain interpolation conditions.

THEOREM 3.2.1. *Suppose that f has continuous partial dervatives up to order k_i at $\mathbf{x}^i$, $i=1, ..., 2^s$. Then the polynomial*

$$p_{\mathbf{n}}(f, \mathbf{x})=\sum_{i=1}^{2^s} \sum_{|\alpha| \leqslant k_i} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} (\eta^i)^\beta$$
$$\times \tilde{D}^\beta f(\mathbf{x}^i)\, \tilde{\phi}^{\mathbf{n}}_{R^n_i \alpha}(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$$

satisfies

$$D^\beta p_{\mathbf{n}}(f, \mathbf{x}^i) = D^\beta f(x^i) \qquad |\beta| \leqslant k, \qquad \textit{for} \quad i = 1, ..., 2^s, \tag{3.2.1}$$

where $\mathbf{n} = (n_1, ..., n_s)$ *with* $n_i \geqslant \max\{k_l + k_j, j \neq l\} + 1, i = 1, ..., s.$

Proof. Let

$$p_{\mathbf{n}, k_i}(\mathbf{x}) = \sum_{|\alpha| \leqslant k_i} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} (\eta^i)^\beta \times \hat{D}^\beta f(\mathbf{x}^i)\, \tilde{\phi}_{R_i^n \alpha}(v_1(\mathbf{x}), ..., v_s(\mathbf{x})),$$

$i = 1, ..., 2^s$. It is obvious that we only need verify that $p_{\mathbf{n},k_1}$ satisfies

$$\tilde{D}^\beta p_{\mathbf{n}, k_1}(\mathbf{x}^1) = \tilde{D}^\beta f(\mathbf{x}^1), \qquad |\beta| \leqslant k_1, \tag{5}$$

and

$$\tilde{D}^\beta p_{\mathbf{n}, k_1}(\mathbf{x}^j) = 0, \qquad |\beta| \leqslant k_j, j = 2, ..., 2^s, \tag{6}$$

since the other polynomials $p_{\mathbf{n},k_i}$ can be treated similarly. To do this, we write

$$\begin{aligned} a_\alpha &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} (\eta^i)^\beta\, \tilde{D}^\beta f(\mathbf{x}^1) \\ &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} (-1)^{|\beta|} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} (\eta^i)^\beta\, \hat{D}^\beta f(\mathbf{x}^1). \end{aligned}$$

By using the inversion formula in Theorem 3.1, we find

$$\begin{aligned} (-1)^{|\beta|} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} (\eta^1)^\beta\, \hat{D}^\beta f(x^1) &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} a_\gamma \\ &= (-1)^{|\beta|} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\beta-\gamma|} a_\gamma \\ &= (-1)^{|\beta|} \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_0, \end{aligned}$$

or

$$\frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} \tilde{D}^\beta f(\mathbf{x}^1) = \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_0.$$

Now if we set $p_{\mathbf{n},k_i}(\mathbf{x}) = \sum_{\alpha \leqslant \mathbf{n}} a_\alpha \tilde{\phi}_\alpha^{\mathbf{n}}(v_1, ..., v_s)$, then

$$\tilde{D}^\beta p_{\mathbf{n},k_i}(\mathbf{x}) = \frac{\mathbf{n}!}{(\mathbf{n}-\beta)} \sum_{\gamma \leqslant \mathbf{n}-\beta} \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_\gamma \tilde{\phi}_\gamma^{\mathbf{n}-\beta}(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$$

by the application of Lemma 2.2.1. Hence, we have

$$\tilde{D}^{\beta} p_{\mathbf{n},k_1}(\mathbf{x}^1) = \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_0$$
$$= \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} \tilde{D}^{\beta} f(\mathbf{x}^1), \qquad |\beta| \leqslant k_1.$$

That is, (5) is verified. To see that (6) also holds, we note that $n_i \geqslant \max\{k_j + k_l\} + 1$, $i = 1, ..., s$, and

$$\tilde{D}^{\beta} p_{\mathbf{n},k_1}(\mathbf{x}^i) = \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_{R_i^n \beta}, \qquad i = 2, ..., 2^s,$$

where we may assume that $p_{\mathbf{n},k_1} = \sum_{|\alpha| \leqslant k_1} a_\alpha \tilde{\phi}_\alpha^{\mathbf{n}}(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$. Since $\eta^i \neq \eta^1 = (1, ..., 1)$, we have $a_{R_i^n \beta} = 0$ for $|R_i^n \beta + \gamma| \geqslant \min(n_i - \beta_i) \geqslant 1 + k_1$ so that

$$\tilde{D}^{\beta} p_{\mathbf{n},k_1}(\mathbf{x}^1) = \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\beta - \gamma|} a_{R_i^n \beta + \gamma}$$
$$= 0.$$

Therefore, the theorem is established.

Let $N_{ij} \in \mathbf{Z}_+$ and $N_i^s = \{(\beta_1, ..., \beta_s) \in \mathbf{Z}_+ : \beta_j \leqslant N_{ij}, j = 1, ..., s\}$, $i = 1, ..., 2^s$. By using an argument similar to that in the proof of the Theorem 3.2.1 we have the following result.

THEOREM 3.2.2. *Suppose that f is sufficiently smooth at each vertex $\mathbf{x}^i$, $i = 1, ..., 2^s$. Then the polynomial*

$$p_{\mathbf{n}}(f, \mathbf{x}) = \sum_{i=1}^{s} \sum_{\beta \in N_i^s} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(\mathbf{n}-\gamma)!}{\mathbf{n}!} (\eta^i) \tilde{D}^{\gamma} f(\mathbf{x}^i) \tilde{\phi}_{R_i^n \beta}(v_1(\mathbf{x}), ..., v_s(\mathbf{x}))$$

satisfies

$$\tilde{D}^{\beta} p_n(f, \mathbf{x}^i) = \tilde{D}^{\beta} f(\mathbf{x}^i), \qquad \beta \in N_i^s, \ i = 1, ..., 2^s, \tag{3.2.2}$$

where $\mathbf{n} = (n_1, ..., n_s) \in \mathbf{Z}_+^s$ with $n_i \geqslant \max\{N_{ji} + N_{ki}, j \neq k\}$, $i = 1, ..., 2^s$.

Of course, the polynomials in Theorems 3.2.1 and 3.2.2 may not unique. We now study the situations when these interpolation problems have unique solutions. We again need a definition of partition of Γ_n^s as follows:

A collection of subsets $N_1^s, ..., N_{2^s}^s \subset \Gamma_{\mathbf{n}}^s$ is said to form a partition of $\Gamma_{\mathbf{n}}^s$ if

(i) $R_i^{\mathbf{n}} N_i^s \cap R_j^{\mathbf{n}} N_j^s = \varnothing$ for $i \neq j$ and

(ii) $\bigcup_{i=1}^{2^s} R_i^{\mathbf{n}} N_i^s = \Gamma_{\mathbf{n}}^s$.

THEOREM 3.2.3. *Suppose that $N_i^s \subset \Gamma_{\mathbf{n}}^s$, $i=1, \ldots, 2^s$, are lower sets and form a partition of $\Gamma_{\mathbf{n}}^s$ Then for any given data $\{f_{i\beta}: \beta \in N_i^s, i=1, \ldots, 2^s\}$, there exists a unique interpolation polynomial $p_{\mathbf{n}} \in \tilde{\pi}_{\mathbf{n}}^s(S)$ that satisfies*

$$\tilde{D}^{\beta} p_{\mathbf{n}}(\mathbf{x}^i) = f_{i\beta}, \qquad \beta \in N_i^s, i=1, \ldots, 2^s. \tag{3.2.3}$$

Moreover, $p_{\mathbf{n}}$ can be formulated as

$$p_{\mathbf{n}} = \sum_{i=1}^{2^s} \sum_{\alpha \in N_i^s} \left(\sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \frac{(\mathbf{n}-\gamma)!}{\mathbf{n}!} (\eta^i)^{\gamma} f_{i\gamma} \right) \tilde{\phi}_{R_i^n \alpha}(v_1(\mathbf{x}), \ldots, v_s(\mathbf{x})). \tag{3.2.4}$$

Proof. From the assumption, the sets $\{a_{\beta}: \beta \in R_i^{\mathbf{n}} N_i^s\}$, $i=1, \ldots, 2^s$, are mutually disjoint. Since

$$\begin{aligned}
\tilde{D}^{\beta} p_{\mathbf{n}}(\mathbf{x}^1) &= \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \Delta_1^{\beta_1} \cdots \Delta_s^{\beta_s} a_0 \\
&= \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\beta-\gamma|} a_{\gamma} \\
&= (-1)^{\beta} \frac{\mathbf{n}!}{(\mathbf{n}-\beta)!} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} a_{\gamma}
\end{aligned}$$

for $\beta \in N_1^s = R_1^{\mathbf{n}} N_1^s$, the inversion formula in Theorem 3.1 gives

$$\begin{aligned}
a_{\beta} &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} (-1)^{|\gamma|} \frac{(\mathbf{n}-\gamma)!}{\mathbf{n}!} \tilde{D}^{\gamma} p_{\mathbf{n}}(\mathbf{x}^1) \\
&= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(\mathbf{n}-\beta)!}{\mathbf{n}!} f_{1\gamma}.
\end{aligned}$$

This quantity is uniquely determined by the given data for $\beta \in N_1^s$, since N_1^s is a lower set. Similarly, $\{a_{\gamma}: \gamma \in R_i^{\mathbf{n}} N_i^s\}$, $i \geqslant 2$, is uniquely determined by $\{f_{i\gamma}: \gamma \in N_i^s\}$. The existence and uniqueness of the interpolation polynomial $p_{\mathbf{n}}$ that satisfies (3.2.3) follow by choosing a_{β} as above; i.e.,

$$a_{R_i^n \beta} = \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(\mathbf{n}-\gamma)!}{\mathbf{n}!} (\eta^i)^{\gamma} f_{i\gamma}, \qquad \beta \in N_i^s, i=1, \ldots, 2^s,$$

which are the coefficients in (3.2.4). Thus, the theorem is established.

From the proof of this theorem, we see that a_{β}, $\beta \in R_i^{\mathbf{n}} N_i^s$, are obtained by using the previous information. Hence, the requirement that N_i^s, $i=1, \ldots, 2^s$, be lower sets in Theorem 3.2.3 can be slightly relaxed, and the resulting theorem will become more applicable. That is, we have the following generalization.

THEOREM 3.2.4. *Suppose that $N_i^s \subset \Gamma_{\mathbf{n}}^s$, $i=1, ..., 2^s$, form a partition of $\Gamma_{\mathbf{n}}^s$ and suppose further that*

(i) *N_1^s is a lower set, and*

(ii) *the union of N_j^s and some subset of $(R_j^{\mathbf{n}})^{-1} (\bigcup_{i=0}^{j-1} R_i^{\mathbf{n}} N_i^s)$ is also a lower set for $j=2, ..., 2s$.*

Then there exists a unique polynomial $p_{\mathbf{n}} \in \tilde{\pi}_{\mathbf{n}}^s(S)$ that satisfies (3.2.3) *for any given data $\{f_{i\beta}: \beta \in N_i^s, i=1, ..., 2^s\}$.*

EXAMPLE 3.2.1. Let $s=2$ and $\mathbf{n}=(3.4)$. Suppose that $N_1^2=\{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)\}$, $N_2^2=\{(0,0), (0,1), (1,0), (1,1)\}$, $N_3^2=\{(0,0), (1,0)\}$, and $N_4^2=\{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2)\}$ (cf. Fig. 3.2.1 for the relationship between N_i^2, $i=1,2,3,4$, and the Bézier net). Theorem 3.2.3 implies that for any given data $\{f_{i\beta}: \beta \in N_i^2\}$ we can find a unique polynomial that interpolates the given data.

EXAMPLE 3.2.2. Let $s=2$,and $\mathbf{n}=(5,5)$. Suppose that $N_1^2=\{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1), (0,2), (1,2), (2,2)\}$, $N_2^2=\{(0,0), (1,0), (0,1), (1,1)\}$, $N_3^2=\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (3,0)\}$, and $N_4^2=\{(0,0), (1,0), (0,1), (1,1), (2,1), (3,1), (0,2), (1,2), (2,2), (3,2), (0,3), (1,3), (2,3)\}$ (cf. Figure 3.2.2 for the relationship between $\{N_i^2: i=1,2,3,4\}$ and the Bézier net). Theorem 3.2.3 implies that for any given data $\{f_{i\beta}: \beta \in N_i^2, i=1,2,3,4\}$, there is a unique polynomial interpolating the given data, although N_4^2 is not a lower set.

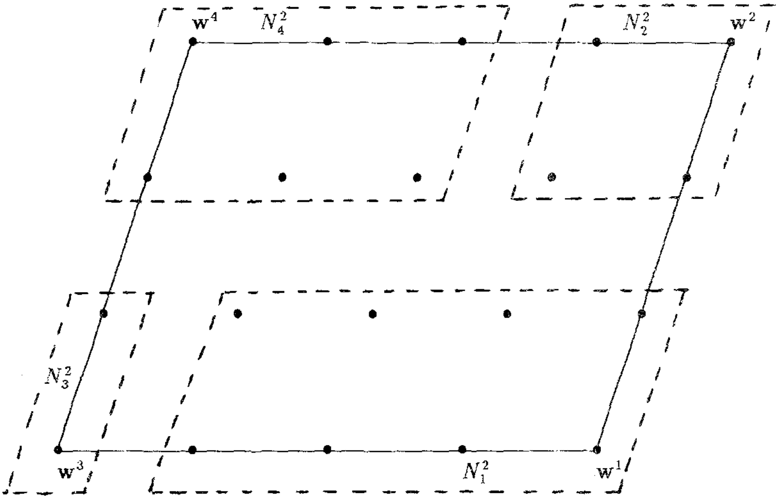

FIGURE 3.2.1

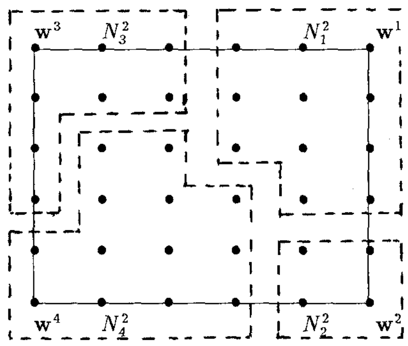

FIGURE 3.2.2

4. SMOOTHNESS CONDITIONS

We next turn our attention to discussing the conditions for two polynomials on adjacent geometric configurations to be joined smoothly together. The geometric configurations under consideration in this section are s-simplices and s-parallelepipeds. Three cases will be studied: two simplices, two parallelepipeds, and a triangle and a parallelogram. Other geometric configurations such as prisms will be studied elsewhere. (See [20].)

4.1. *The Simplex Case*

Suppose that

$$S_1 = \langle \mathbf{x}^0, ..., \mathbf{x}^s \rangle$$

and

$$S_2 = \langle \mathbf{x}^0, ..., \mathbf{x}^k, \mathbf{y}^{k+1}, ..., \mathbf{y}^s \rangle$$

are two s-simplices in $\mathbf{R}^s$ and $T = \langle \mathbf{x}^0, ..., \mathbf{x}^k \rangle$ is a k-simplex which is a common facet of S_1 and S_2, where $0 \leqslant k < s$. Let F be defined on $S_i \cup S_2$ by

$$F(\mathbf{x})|_{S_1} = P_n(\mathbf{x}) = \sum_{|\alpha|=n} = \sum_{|\alpha|=n} a_\alpha \phi_\alpha^n(\lambda_0(\mathbf{x}), ..., \lambda_s(\mathbf{x})),$$

where $\mathbf{x} = \sum_{i=0}^s \lambda_i(\mathbf{x})\, \mathbf{x}^i$ with $\sum_{i=0}^s \lambda_i(\mathbf{x}) \equiv 1$ and

$$F(\mathbf{x})|_{S_2} = \hat{P}_n(\mathbf{x}) = \sum_{|\alpha|=n} \hat{a}_\alpha \phi_\alpha^n(v_0(\mathbf{x}), ..., v_s(\mathbf{x})),$$

where $\mathbf{x} = \sum_{i=0}^k v_i(\mathbf{x})\, \mathbf{x}^i + \sum_{i=k+1}^s v_i(\mathbf{x})\, \mathbf{y}^i$ with $\sum_{i=0}^s v_i(\mathbf{x}) \equiv 1$.

Write $\mathbf{y}^j = \sum_{i=0}^{s} c_{ji} \mathbf{x}^i, j = k+1, ..., s$. We have

THEOREM 4.1.1. *Suppose that S_1 and S_2 are two s-simplices such that $T = S_1 \cap S_2$ is a k-simplex in $\mathbf{R}^s$. Then $F \in C^r(S_1 \cup S_2)$ if and only if the conditions*

$$\begin{aligned} &\Delta_{k+1,0}^{\gamma_{k+1}} \cdots \Delta_{s0}^{\gamma_s} \hat{a}_{(\alpha_0, ..., \alpha_k, 0, ..., 0)} \\ &\quad = \left(\sum_{i=1}^{s} c_{k+1,i} \Delta_{i0} \right)^{\gamma_{k+1}} \cdots \left(\sum_{i=0}^{s} c_{si} \Delta_{i0} \right)^{\gamma_s} a_{(\alpha_0, ..., \alpha_k, 0, ..., 0)} \end{aligned} \tag{4.1.1}$$

hold for $0 \leqslant \gamma_{k+1} + \cdots + \gamma_s \leqslant r, \alpha_0 + \cdots + \alpha_k + \gamma_{k+1} + \cdots + \gamma_s = n$.

Proof. If $r = 0$, it is clear that $F \in C^0(S_1 \cup S_2)$ if and only if (4.1.1) holds for $\alpha_0 + \cdots + \alpha_k = n$ since two polynomials agree on T if and only if their Bézier coefficients on T are equal. Suppose that $F \in C^r(S_1 \cup S_2)$, where $r \geqslant 1$. Since $\mathbf{y}^j - \mathbf{x}^0 = \sum_{i=0}^{s} c_{ji}(\mathbf{x}^i - \mathbf{x}^0)$, it follows that

$$(D_{\mathbf{y}^j - \mathbf{x}^0})^{\beta_j} \hat{P} |_T = \left(\sum_{i=1}^{s} c_{ki} D_{i0} \right)^{\beta_j} P \bigg|_T, \qquad \beta_j \geqslant 0.$$

Observing that

$$(D_{\mathbf{y}^j - \mathbf{x}^0})^{\beta_j} \hat{P} |_T = \frac{n!}{(n-\beta_j)!} \sum_{|\alpha| = n - \beta_j} \Delta_{j0}^{\beta_j} \hat{a}_\alpha \phi_\alpha^{n-\beta_j} \bigg|_T$$

and

$$\left(\sum_{i=0}^{s} c_{ji} D_{i0} \right)^{\beta_j} P \bigg|_T = \frac{n!}{(n-\beta_j)!} \sum_{|\alpha| = n - \beta_j} \left(\sum_{i=0}^{s} c_{ji} \Delta_{i0} \right)^{\beta_j} a_\alpha \phi_\alpha^{n-\beta_j} \bigg|_T,$$

we have the equivalent conditions

$$\Delta_{j0}^{\beta_j} \hat{a}_{(\alpha_0, ..., \alpha_k, 0, ..., 0)} = \left(\sum_{i=0}^{s} c_{ji} \Delta_{i0} \right)^{\beta_j} a_{(\alpha_0, ..., \alpha_k, 0, ..., 0)}$$

for $\alpha_0 + \cdots + \alpha_k = n - \beta_j, j = k+1, ..., s$. Similarly, the conditions in (4.1.1) that follow from equating the mixed derivatives are also obtained easily.

On the other hand, suppose that (4.1.1) holds for $0 \leqslant \beta_{k+1} + \cdots + \beta_s \leqslant r$, $\alpha_0 + \cdots + \alpha_k + \beta_{k+1} + \cdots + \beta_s = n$, where $r \geqslant 0$. It follows that

$$\begin{aligned} &(D_{\mathbf{y}^{k+1} - \mathbf{x}^0})^{\beta_{k+1}} \cdots (D_{\mathbf{y}^s - \mathbf{x}^0})^{\beta_s} \hat{P}_n |_T \\ &\quad = \left(\sum_{i=0}^{s} c_{k+1,i} D_{i0} \right)^{\beta_{k+1}} \cdots \left(\sum_{i=0}^{s} c_{si} D_{i0} \right)^{\beta_s} P_n \bigg|_T \end{aligned}$$

for all $\beta_{k+1}+\cdots+\beta_s\leqslant r$. Consequently, we have

$$\prod_{j=0}^{k}(D_{j0})^{\beta_j}(D_{\mathbf{y}^{k+1}-\mathbf{x}^0})^{\beta_{k+1}}\cdots(D_{\mathbf{y}^s-\mathbf{x}^0})^{\beta_s}\hat{P}_n\Big|_T$$
$$=\prod_{j=0}^{k}(D_{j0})^{\beta_j}\left(\sum_{i=0}^{s}c_{k+1,i}D_{i0}\right)^{\beta_k}\cdots\left(\sum_{i=0}^{s}c_{si}D_{i0}\right)^{\beta_s}P_n\Big|_T$$

for $\beta_1+\cdots+\beta_k\leqslant r-\beta_{k+1}-\cdots-\beta_s$. This implies that $F\in C^r(S_1\cup S_2)$, and the proof of the theorem is completed.

It should also be noted that the smoothness conditions can be formulated by using the information of F at one vertex. More precisely, we have the following result.

THEOREM 4.1.2. *$F\in C^r(S_1\cup S_2)$ if and only if*

$$D_{10}^{\gamma_1}\cdots D_{k0}^{\gamma_k}(D_{\mathbf{y}^{k+1}-\mathbf{x}^0})^{\beta_{k+1}}\cdots(D_{\mathbf{y}^s-\mathbf{x}^0})^{\beta_s}\hat{P}_n(\mathbf{x}^0)$$
$$=D_{10}^{\gamma_1}\cdots D_{k0}^{\gamma_k}\left(\sum_{i=0}^{s}c_{k+1,i}D_{i0}\right)^{\beta_{k+1}}\cdots\left(\sum_{i=0}^{s}c_{s0}D_{i0}\right)^{\beta_s}P_n(\mathbf{x}^0)\qquad(4.1.2)$$

for all $\gamma_1+\cdots+\gamma_k\leqslant n-(\beta_{k+1}+\cdots+\beta_s)$ and $\beta_{k+1}+\cdots+\beta_s\leqslant r$.

Proof. If $F\in C^r(S_1\cup S_2)$, then

$$(D_{\mathbf{y}^{k+1}-\mathbf{x}^0})^{\beta_{k+1}}\cdots(D_{\mathbf{y}^s-\mathbf{x}^0})^{\beta_s}\hat{P}_n|_T$$
$$=\left(\sum_{i=0}^{s}c_{k+1,i}D_{i0}\right)^{\beta_{k+1}}\cdots\left(\sum_{i=0}^{s}c_{si}D_{i0}\right)^{\beta_s}P_n\Big|_T$$

for $\beta_{k+1}+\cdots+\beta_s=l$, $l=0,\ldots,r$, and hence (4.1.2) holds for all $\gamma_1+\cdots+\gamma_k\leqslant n-(\beta_{k+1}+\cdots+\beta_s)$ and $\beta_{k+1}+\cdots+\beta_s\leqslant r$.

On the other hand, it is clear that (4.1.2) is equivalent to the condition

$$\Delta_{10}^{\gamma_1}\cdots\Delta_{k0}^{\gamma_k}\Delta_{k+1,0}^{\beta_{k+1}}\cdots\Delta_{s0}^{\beta_s}\hat{a}_{(\alpha_0,0,\ldots,0)}$$
$$=\Delta_{10}^{\gamma_1}\cdots\Delta_{k0}^{\gamma_k}\left(\sum_{i=0}^{s}c_{k+1,i}\Delta_{i0}\right)^{\beta_{k+1}}\cdots\left(\sum_{i=0}^{s}c_{si}\Delta_{i0}\right)^{\beta_s}a_{(\alpha_0,0,\ldots,0)}$$

for $\gamma_1+\cdots+\gamma_k\leqslant n-\beta_{k+1}-\cdots-\beta_s$, $\beta_{k+1}+\cdots+\beta_s\leqslant r$, where $\alpha_0=n-(\gamma_1+\cdots+\gamma_k+\beta_{k+1}+\cdots+\beta_s)$. Since Δ_{i0}, $i=1,\ldots,s$, are differences, it follows from the inversion formula in Theorem 3.1 that

$$\Delta_{k+1,0}^{\beta_{k+1}}\cdots\Delta_{s0}^{\beta_s}\hat{a}_{(\alpha_0,\ldots,\alpha_k,0,\ldots,0)}$$
$$=\left(\sum_{i=1}^{s}c_{k+1,i}\Delta_{i0}\right)^{\beta_{k+1}}\cdots\left(\sum_{i=0}^{s}c_{si}\Delta_{i0}\right)^{\beta_s}a_{(\alpha_0,\ldots,\alpha_k,0,\ldots,0)}$$

for $\beta_{k+1}+\cdots\beta_s\leqslant r$, $\alpha_0+\cdots+\alpha_k=n-(\beta_{k+1}+\cdots+\beta_s)$. By Theorem 4.1.1, we have $F\in C^r(S_1\cup S_2)$, which completes the proof of the theorem.

In fact, the idea in the above proof can yield a little more. We need the following notation. Let $M_{n,r,k}=\{\alpha\in\mathbf{Z}^{s+1}: |\alpha|=n, \alpha_{k+1}+\cdots+\alpha_s\leqslant r\}$ and write

$$\mathbf{y}^i-\mathbf{x}^j=\sum_{\substack{l=0\\ l\neq j}}^{s} c_l^{ij}(\mathbf{x}^l-\mathbf{x}^j), \qquad i=k+1, ..., s \quad \text{and} \quad j=0, ..., k.$$

Let

$$\hat{D}_j^\beta := \prod_{\substack{i=0\\ i\neq j}}^{k} D_{ij}^{\beta_i} \prod_{i=k+1}^{s} (D_{\mathbf{y}^i-\mathbf{x}^j})^{\beta_i}$$

and

$$\bar{D}_j^\beta := \prod_{\substack{i=0\\ i\neq j}}^{k} D_{ij}^{\beta_i} \prod_{i=k+1}^{s} \left(\sum_{l=0}^{s} c_l^{ij} D_{lj}\right)^{\beta_i},$$

where $j=0, ..., k$.

Then we have the following generalization of Theorem 4.1.2.

THEOREM 4.1.3. *Suppose that* M_i^{s+1}, $i=0, ..., k$, *are mutually disjoint subsets of* $M_{n,r,k}$ *and* $\bigcup_{i=0}^{k} M_i^{s+1}=M_{n,r,k}$. *Furthermore, suppose that* $c_j M_j^{s+1}$ *is a lower set for* $j=0, ..., k$. *Then* $F\in C^r(S_1\cup S_2)$ *if and only if*

$$\hat{D}_j^\beta \hat{P}_n(\mathbf{x}^j)=\bar{D}_j^\beta P_n(\mathbf{x}^j) \qquad (4.1.3)$$

for $\beta\in c_j M_j^{s+1}, j=0, ..., k$.

The proof is similar to that of Theorem 4.1.2. Recall that the operator c_j was defined in the beginning of the last section.

Remark 1. One consequence of the above theorem is that it is not necessary to use normal derivatives to ensure $F\in C^r(S_1\cup S_2)$.

Remark 2. Different versions of the smoothness conditions on polynomials over adjacent simplices have been studied and be found in [15, 11, 5, 16, 19]. Here, generalized versions of our earlier work in [11] were presented. In the following, we will establish the relationship between our results and those of the others.

THEOREM 4.1.4. *$F \in C^r(S_1 \cup S_2)$ if and only if*

$$\begin{aligned}
&\hat{a}_{(\alpha_0, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_s)} \\
&= \left(\sum_{i=0}^{s} c_{k+1,i} S_i\right)^{\beta_{k+1}} \cdots \left(\sum_{i=0}^{s} c_{si} S_i\right)^{\beta_s} a_{(\alpha_0, \dots, \alpha_k, 0, \dots, 0)} \\
&= \sum_{\substack{|\gamma_j| = \beta_j \\ \gamma_j \in \mathbf{Z}^{s+1} \\ j = k+1, \dots, s}} a_{(\alpha_0, \dots, \alpha_k, 0, \dots, 0) + \gamma_{k+1} + \cdots + \gamma_s} \\
&\quad \times \phi^{\beta_{k+1}}_{\gamma_{k+1}}(c_{k+1,0}, \dots, c_{k+1,s}) \cdots \phi^{\beta_s}_{\gamma_s}(c_{s,0}, \dots, c_{ss}) \qquad (4.1.4)
\end{aligned}$$

for $\beta_{k+1} + \cdots + \beta_s \leqslant r$, $\alpha_0 + \cdots + \alpha_k + \beta_{k+1} + \cdots + \beta_s = n$.

Proof. Since for $\eta_{k+1} + \cdots + \eta_s \leqslant r$,

$$\begin{aligned}
&(-1)^{|\eta_{k+1} + \cdots + \eta_s|} \Delta_{j0}^{\eta_{k+1}} \cdots \Delta_{s0}^{\eta_s} \hat{a}_{(n - \eta_{k+1} - \cdots - \eta_s - \alpha_1 - \cdots - \alpha_k, \alpha_1, \dots, \alpha_s)} \\
&= \sum_{\substack{\beta_j \leqslant \eta_j \\ j = k+1, \dots, s}} \binom{\eta_{k+1}}{\beta_{k+1}} \cdots \binom{\eta_s}{\beta_s} (-1)^{|\beta_{k+1} + \cdots + \beta_s|} \hat{a}_{(\alpha_0, \alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_s)},
\end{aligned}$$

the inversion formula in Theorem 3.1 can be applied to yield

$$\begin{aligned}
\hat{a}_{(\alpha_0, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_s)} = &\sum_{\substack{\eta_i \leqslant \beta_i \\ i = k+1, \dots, s}} \binom{\beta_{k+1}}{\eta_{k+1}} \cdots \binom{\beta_s}{\eta_s} \Delta_{k+1,0}^{\eta_{k+1}} \cdots \Delta_{s0}^{\eta_s} \\
&\times \hat{a}_{(n - \eta_{k+1} - \cdots - \eta_s - \alpha_1 - \cdots - \alpha_k, \alpha_1, \dots, \alpha_k, 0, \dots, 0)}.
\end{aligned}$$

By Theorem 4.1.1, we have

$$\begin{aligned}
&\hat{a}_{(\alpha_0, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_s)} \\
&= \sum_{\substack{\eta_i \leqslant \beta_i \\ i = k+1, \dots, s}} \binom{\beta_{k+1}}{\eta_{k+1}} \cdots \binom{\beta_s}{\eta_s} \left(\sum_{i=1}^{s} c_{k+1,i} \Delta_{i0}\right)^{\eta_{k+1}} \cdots \left(\sum_{i=1}^{s} c_{si} \Delta_{i0}\right)^{\eta_s} \\
&\quad \times a_{(n - \eta_{k+1} - \cdots - \eta_s - \alpha_1 - \cdots - \alpha_k, \alpha_1, \dots, \alpha_k, 0, \dots, 0)} \\
&= \sum_{\substack{\eta_i \leqslant \beta_i \\ i = k+1, \dots, s}} \binom{\beta_{k+1}}{\eta_{k+1}} \cdots \binom{\beta_s}{\eta_s} \left(\sum_{i=1}^{s} c_{k+1,i} \Delta_{i0}\right)^{\eta_{k+1}} \cdots \left(\sum_{i=1}^{s} c_{si} \Delta_{i0}\right)^{\eta_s} \\
&\quad \times (E_0)^{\beta_{k+1} + \cdots + \beta_s - \eta_{k+1} - \cdots - \eta_s} \\
&\quad \times a_{(n - \beta_{k+1} - \cdots - \beta_s - \alpha_1 - \cdots - \alpha_k, \alpha_1, \dots, \alpha_k, 0, \dots, 0)}
\end{aligned}$$

$$= \left(E_0 + \sum_{i=1}^{s} c_{k+1,i} \Delta_{i0} \right)^{\beta_{k+1}} \cdots \left(E_0 + \sum_{i=1}^{s} c_{si} \Delta_{i0} \right)^{\beta_s}$$

$$\times a_{(n-\beta_{k+1}-\cdots-\beta_s-\alpha_1-\cdots-\alpha_k,\, \alpha_1, \ldots, \alpha_k, 0, \ldots, 0)}$$

$$= \left(\sum_{i=0}^{s} c_{k+1,i} E_i \right)^{\beta_{k+1}} \cdots \left(\sum_{i=0}^{s} c_{si} E_i \right)^{\beta_s}$$

$$\times a_{(n-\beta_{k+1}-\cdots-\beta_s-\alpha_1-\cdots-\alpha_k,\, \alpha_1, \ldots, \alpha_k, 0, \ldots, 0)}.$$

Therefore, the theorem is established.

We note, in particular, that when $k = s-1$, we have

$$\hat{a}_{(\alpha_0, \ldots, \alpha_{s-1}, l)} = \left(\sum_{i=0}^{s} c_{si} s_i \right)^{l} a_{(\alpha_0, \ldots, \alpha_{s-1}, 0)}$$

$$= \sum_{|\gamma| = l} a_{(\alpha_0, \ldots, \alpha_s - 1, 0) + \gamma} \phi^l_\gamma (c_{s0}, \ldots, c_{ss})$$

which can be seen to be the same as the versions in [15] and [19].

EXAMPLE 4.1.1. Let $s=2$ and $S = \langle V^0, V^1, V^3 \rangle$, $\hat{S} = \langle V^0, U^1, U^2 \rangle$ be two 2-simplices with a common vertex V^0. Let the polynomials P_3 on S and $\hat{P}_3$ on $\hat{S}$ be expressed by using their Bézier nets as shown in Fig. 4.1.1. Write $U^i = \alpha_i V^0 + \beta_i V^1 + \gamma_i V^3$, $\alpha_i + \beta_i + \gamma_i = 1$, $i = 1, 2$. Define F by $F|_S = P_3$ and $F|_{\hat{S}} = \hat{P}_3$. Then

(1) $F \in C(S \cup \hat{S})$ if and only if $a = k$;

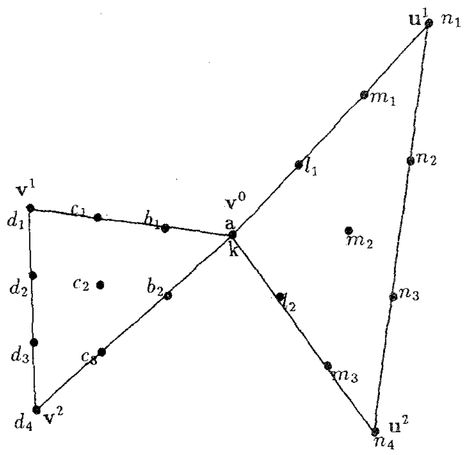

FIGURE 4.1.1

(2) $F \in C^1(S \cup \hat{S})$ if and only if $a = k$,

$$l_1 = \alpha_1 a + \beta_1 b_1 + \gamma_1 b_2,$$

and

$$l_2 = \alpha_2 a + \beta_2 b_1 + \gamma_2 b_2;$$

(3) $F \in C^2(S \cup \hat{S})$ if and only if, in addition to the above relations,

$$m_1 = \alpha_1 l_1 + \beta_1(\alpha_1 b_1 + \beta_1 c_1 + \gamma_1 c_2) + \gamma_1(\alpha_1 b_2 + \beta_1 c_2 + \gamma_1 c_3),$$

$$m_2 = \alpha_2 l_1 + \beta_2(\alpha_1 b_1 + \beta_1 c_1 + \gamma_1 c_2) + \gamma_2(\alpha_1 b_2 + \beta_1 c_2 + \gamma_1 c_3),$$

and

$$m_3 = \alpha_2 l_2 + \beta_2(\alpha_2 b_1 + \beta_2 c_1 + \gamma_2 c_2) + \gamma_2(\alpha_2 b_2 + \beta_2 c_2 + \gamma_2 c_3).$$

The geometric interpretation of the smoothness conditions is interesting. See Fig. 4.1.2.

EXAMPLE 4.1.2. Let $s = 2$ and $n = 3$. Suppose that $S = \langle V^0, V^1, V^2 \rangle$ and $\hat{S} = \langle V^0, V^1, U^2 \rangle$ are two 2-simplices and P_3 and $\hat{P}_3$ are two polynomials of total degree $\leqslant 3$ whose Bézier nets are displayed on their domains S and $\hat{S}$, respectively (cf. Fig. 4.1.3). Write $U^2 = \alpha V^0 + \beta V^1 + \gamma V^2$, where $\alpha + \beta + \gamma = 1$. Then

(1) $F \in C(S \cup \hat{S})$ if and only if

$$a_i = l_i, \qquad i = 1, 2, 3, 4;$$

(2) $F \in C^1(S \cup \hat{S})$ if and only if (1) is satisfied and

$$m_i = \alpha a_{i+1} + \beta a_i + \gamma b_i, \qquad i = 1, 2, 3;$$

(3) $F \in C^2(S \cup \hat{S})$ if and only if (2) is satisfied and

$$n_i = \alpha m_{i+1} + \beta m_i + \gamma(\alpha b_{i+1} + \beta b_i + \gamma c_i), \qquad i = 1, 2;$$

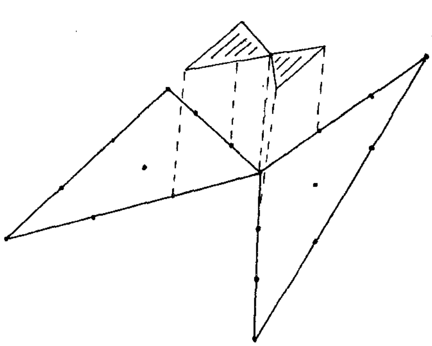

FIGURE 4.1.2

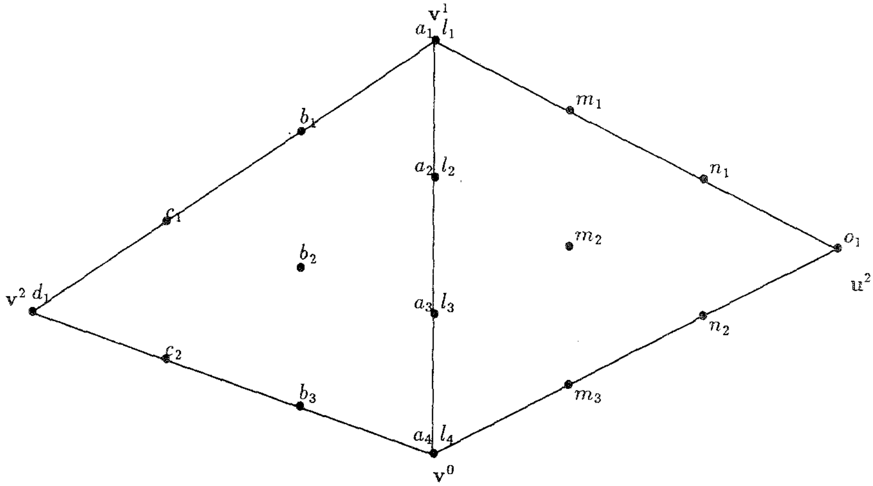

FIGURE 4.1.3

and

(4) $F \in C^3(S \cup \hat{S})$ if and only if (3) is satisfied and

$$o_1 = \alpha n_2 + \beta n_1 + \gamma(\alpha(\alpha b_3 + \beta b_2 + \gamma c_2) + \beta(\alpha b_2 + \beta b_1 + \gamma c_1) + \gamma(\alpha c_2 + \beta c_1 + \gamma d)).$$

The geometric interpretation of the C^3 smoothness conditions is shown in Figs. 4.14a, 4.14b, 4.14c.

EXAMPLE 4.1.3. Let $s=2$ and $n=3$. Write $U^2 - V^0 = \beta(V^1 - V^0) + \gamma(V^2 - V^0)$ and $U^2 - V^1 = \alpha(V^0 - V^1) + \gamma(V^2 - V^1)$. Then $F \in C^1(S \cup \hat{S})$ if and only if

$$D^{\alpha_1}_{U^2 - V^0} D^{\alpha_2}_{V^1 - V^0} \hat{P}_3(V^0) = (\beta D_{V^1 - V^0} + \gamma D_{V^2 - V^0})^{\alpha_1} D^{\alpha_2}_{V^1 - V^0} P_3(V^0)$$

and

$$D^{\beta_1}_{U^2 - V^1} D^{\beta_2}_{V^0 - V^1} \hat{P}_3(V^0) = (\beta D_{V^0 - V^1} + \gamma D_{V^2 - V^1})^{\beta_1} D^{\beta_2}_{V^0 - V^1} P_3(V^0)$$

for $(\alpha_1, \alpha_2) \in \{(0,0), (1,0), (0,1), (1,1)\}$ and $(\beta_1, \beta_2) \in \{(0,0), (1,0), (0,1)\}$, which are both lower sets. Of course, there are many other choices of such sets of (α_1, α_2) and (β_1, β_2).

4.2. *The Parallelepiped Case*

Suppose that

$$S = \langle \mathbf{w}^1, ..., \mathbf{w}^{2^s} \rangle$$

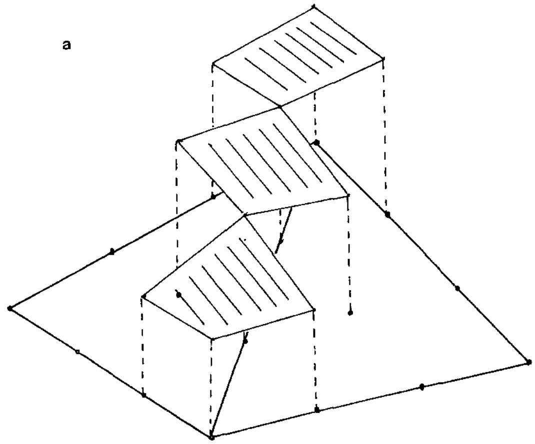

FIGURE 4.1.4a

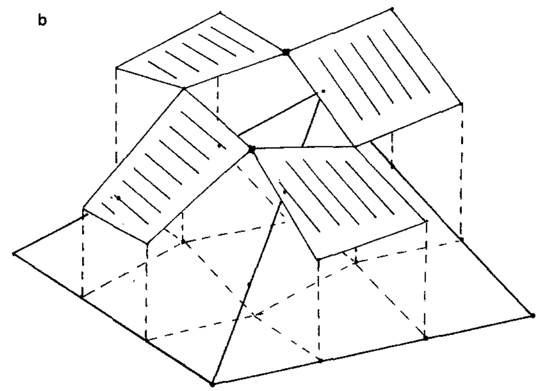

FIGURE 4.1.4b

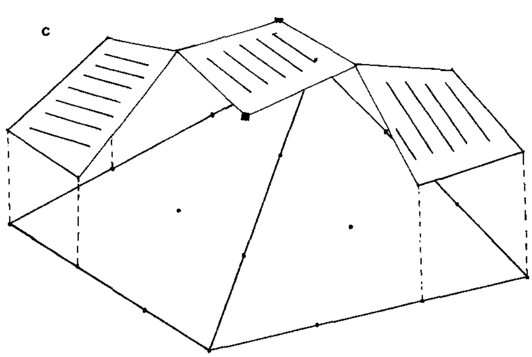

FIGURE 4.1.4c

and

$$\hat{S} = \langle \mathbf{w}^1, \ldots, \mathbf{w}^{2^k}, \mathbf{u}^{2^k+1}, \ldots, \mathbf{u}^{2^s} \rangle$$

are two s-parallelepipeds in $\mathbf{R}^s$ with a common facet $T = \langle w^1, \ldots, w^{2^k} \rangle$ which is a k-parallelepiped, where $0 \leqslant k < s$. Let $(v_1(\mathbf{x}), \ldots, v_s(\mathbf{x}), \ldots, v_s(\mathbf{x}))$ and $(\mu_1(\mathbf{x}), \ldots, \mu_s(\mathbf{x}))$ be the barycentric coordinates of $\mathbf{x}$ with respect to S and $\hat{S}$, respectively. Without loss of generality, by some rearrangement if necessary, we assume that

$$v_i(\mathbf{w}^1) = 0 = \mu_i(\mathbf{w}^1), \qquad i = 1, \ldots, s,$$

$$v_i(\mathbf{w}^{i+1}) = 1 = \mu_i(\mathbf{w}^{i+1}), \qquad i = 1, \ldots, k,$$

and

$$v_{k+j}(\mathbf{w}^{2^k+j}) = 1 = \mu_{k+j}(\mathbf{u}^{2^k+j}), \qquad j = 1, \ldots, s-k.$$

(See Fig. 4.2.1 for reference).

For any polynomial $p_\sigma = \sum_{\alpha \leqslant \sigma} a_\alpha^\sigma \tilde{\phi}_\alpha^\sigma$, we define a degree raising operator R_j, $1 \leqslant j \leqslant s$, by

$$R_j a_\alpha^\sigma = \frac{\alpha_j}{n_j} a_{\alpha - e^j}^\sigma + \left(1 - \frac{\alpha_j}{n_j}\right) a_\alpha^\sigma := a_\alpha^{\sigma + e^j}$$

and

$$R_j^l a_\sigma^\sigma = \sum_{i=0}^{\alpha_j} a_{\alpha + (i - \alpha_j)e^j}^{\sigma + le^j} \frac{\binom{n_j}{i}\binom{l}{\alpha_j - i}}{\binom{n+l}{\alpha_j}},$$

where $\sigma = (n_1, \ldots, n_s)$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbf{Z}_+^s$. Clearly,

$$p_\sigma(\mathbf{x}) = \sum_{\alpha \leqslant \sigma + e^j} R_j a_\alpha^\sigma \tilde{\phi}_\alpha^{\sigma + e^j}.$$

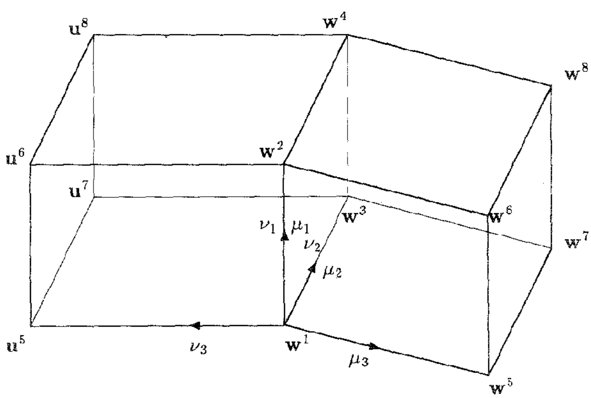

FIGURE 4.2.1

Suppose that F is a piecewise polynomial function defined on $S \cup \hat{S}$ by $F|_S = p_{\mathbf{n}}$ and $F|_{\hat{S}} = \hat{p}_{\mathbf{n}}$, where

$$p_{\mathbf{n}}(\mathbf{x}) = \sum_{\alpha \leqslant \mathbf{n}} a_\alpha \tilde{\phi}_\alpha^{\mathbf{n}}(v_1(\mathbf{x}), ..., v_s(\mathbf{x})),$$

$$\hat{p}_{\mathbf{n}}(\mathbf{x}) = \sum_{\beta \leqslant \mathbf{n}} \hat{a}_\alpha \tilde{\phi}_\alpha^{\mathbf{n}}(v_1(\mathbf{x}), ..., \mu_s(\mathbf{x})),$$

and $\mathbf{n} = (n, ..., n) \in \mathbf{Z}_+^s$. Let

$$D_j = \hat{D}_j = D_{\mathbf{w}^{j+1} - \mathbf{w}^1}, \qquad j = 1, ..., k,$$

$$D_j = D_{\mathbf{w}^{2^k + j - k} - \mathbf{w}^1},$$

and

$$\hat{D}_j = D_{\mathbf{u}^{2^k + j - k} - \mathbf{w}^1}, \qquad j = k+1, ..., s.$$

Choose $\mathbf{c}^j = (c_{j1}, ..., c_{js})$, $j = k+1, ..., s$, such that

$$\mathbf{u}^{2^k + j - k} - \mathbf{w}^1 = \sum_{i=1}^{k} c_{ji}(\mathbf{w}^{i+1} - \mathbf{w}^1) + \sum_{i=k+1}^{s} c_{ji}(\mathbf{w}^{2^k + j - k} - \mathbf{w}^1).$$

We are now ready to state and establish the following theorems.

THEOREM 4.2.1. *Let $r = 0, 1, ...$. Then $F \in C^r(S \cup \hat{S})$ if and only if*

$$\Delta_{k+1}^{\beta_{k+1}} \cdots \Delta_s^{\beta_s} \hat{a}_\gamma = \sum_{|\alpha| = |\beta|} b_\alpha \frac{(\mathbf{n} - \beta)!}{(\mathbf{n} - \alpha)!} \Delta_1^{\alpha_1} \cdots \Delta_s^{s} R_1^{\alpha_1} \cdots R_k^{\alpha_k} a_\gamma^{\sigma(\alpha)} \quad (4.2.1)$$

for $\beta = \beta_{k+1} e^{k+1} + \cdots + \beta_s e^s$ with $|\beta| \leqslant r$ and $\gamma \leqslant ne^1 + \cdots + ne^k$, where $a_\gamma^{\mathbf{n}} = a_\gamma$, $\sigma(\alpha) = (n - \alpha_1, ..., n - \alpha_k, n, ..., n)$, and

$$b_\alpha = \sum_{\substack{\eta^{k+1} + \cdots + \eta^s = \alpha \\ |\eta^j| = \beta_j, j = k+1, ..., s}} \prod_{j=k+1}^{s} \frac{\beta_j! \, (c^j)^{\eta^j}}{\eta^j!}, \qquad |\alpha| = |\beta|.$$

Proof. First, note that

$$\hat{D}_j = \sum_{i=1}^{s} c_{ji} D_i, \qquad j = k+1, ..., s$$

and

$$\hat{D}_{k+1}^{\beta_{k+1}} \cdots \hat{D}_s^{\beta_s} = \prod_{j=k+1}^{s} \left(\sum_{i=1}^{s} c_{ji} D_i \right)^{\beta_j}$$

$$= \prod_{j=k+1}^{s} \sum_{|\gamma| = \beta_j} \frac{\beta_j!}{\gamma!} c_j^\gamma D^\gamma$$

$$= \sum_{|\alpha| = \beta_{k+1} + \cdots + \beta_s} b_\alpha D^\alpha,$$

where $D^\alpha=(D_1,...,D_s)^\alpha=D_1^{\alpha_1}\cdots D_s^{\alpha_s}$. Hence, for any $\tilde{p}_n\in\tilde{\pi}_n^s(S)$,

$$\hat{D}_{k+1}^{\beta_{k+1}}\cdots D_s^{\beta_s}\tilde{p}(\mathbf{x})=\frac{\mathbf{n}!}{(\mathbf{n}-\beta)!}\sum_{\gamma\leqslant\mathbf{n}-\beta}\Delta_{k+1}^{\beta_{k+1}}\cdots\Delta_s^{\beta_s}\hat{a}_\gamma\tilde{\phi}_\gamma^{\mathbf{n}-\beta}(\mu_1(\mathbf{x}),...,\mu_s(\mathbf{x})),$$

where $\beta=\beta_{k+1}e^{k+1}+\cdots+\beta_s e^s$. Consequently,

$$\hat{D}_{k+1}^{\beta_{k+1}}\cdots D_s^{\beta_s}\tilde{p}_{\mathbf{n}}(\mathbf{x})|_T=\frac{\mathbf{n}!}{(\mathbf{n}-\beta)!}\sum_{\substack{\gamma=(\gamma_1,...,\gamma_k,0,...,0)\\ \gamma_i\leqslant n,\ i=1,...,k}}\Delta_{k+1}^{\beta_{k+1}}\cdots\Delta_s^{\beta_s}$$
$$\times\tilde{a}_\gamma\tilde{\phi}_\gamma^{ne^1+\cdots+ne^k}(\mu_1(\mathbf{x}),...,\mu_k(\mathbf{x}),0,...,0).$$

On the other hand, for $\alpha=(\alpha_1,...,\alpha_s)\in\mathbf{Z}_+^s$,

$$D^\alpha p_{\mathbf{n}}(\mathbf{x})|_T=\frac{\mathbf{n}!}{(\mathbf{n}-\alpha)!}\sum_{\gamma\leqslant\mathbf{n}-\alpha}\Delta_1^{\alpha_1}\cdots\Delta_s^{\alpha_s}a_\gamma\tilde{\phi}_\gamma^{\mathbf{n}+\alpha}(v(\mathbf{x}),...,v_s(\mathbf{x}))\Big|_T$$
$$=\frac{\mathbf{n}!}{(\mathbf{n}-\alpha)!}\sum_{\gamma\leqslant\mathbf{n}-\alpha+\tilde{\alpha}}\Delta_1^{\alpha_1}\cdots\Delta_s^{\alpha_s}R_1^{\alpha_1}\cdots R_k^{\alpha_k}$$
$$\times a_\gamma^{\sigma(\alpha)}\tilde{\phi}_\gamma^{\mathbf{n}-\alpha+\tilde{\alpha}}(v_1(\mathbf{x}),...,v_s(\mathbf{x}))\Big|_T$$
$$=\frac{\mathbf{n}!}{(\mathbf{n}-\alpha)!}\sum_{\gamma\leqslant ne^1+\cdots+ne^k}\Delta_1^{\alpha_1}\cdots\Delta_s^{\alpha_s}$$
$$\times R_1^{\alpha_1}\cdots R_k^{\alpha_k}\tilde{a}_\gamma^{\sigma(\alpha)}\tilde{\phi}_\gamma^{ne^1+\cdots+ne^k}(v_1(\mathbf{x}),...,v_k(\mathbf{x}),0,...,0),$$

where we have used the degree raising operator.

Therefore, $F\in C^r(S\cup\tilde{S})$ if and only if

$$\hat{D}_{k+1}^{\beta_{k+1}}\cdots D_s^{\beta_s}\tilde{p}_{\mathbf{n}}(\mathbf{x})|_T=\sum_{|\alpha|=\beta_{k+1}+\cdots+\beta_s}b_\alpha D^\alpha p_{\mathbf{n}}(\mathbf{x})\Big|_T.$$

Since $\tilde{\phi}_\gamma^{ne^1+\cdots+ne^k}(v_1(\mathbf{x}),...,v_k(\mathbf{x}),0,...,0)$, $\gamma\leqslant ne^1+\cdots+ne^k$, are linearly independent, (4.2.1) follows immediately. Thus, the proof is established.

As a consequence of Theorem 4.2.1, we have

THEOREM 4.2.2. *Let F, S, $\hat{S}$, and T be defined as in Theorem 4.2.1. Then $F\in C^r(S\cup\hat{S})$ if and only if*

$$\hat{D}_1^{\beta_1}\cdots\hat{D}_k^{\beta_k}\hat{D}_{k+1}^{\beta_{k+1}}\cdots\hat{D}_s^{\beta_s}\hat{p}_{\mathbf{n}}(\mathbf{w}^1)$$
$$=D_1^{\beta_1}\cdots D_k^{\beta_k}\left(\sum_{i=1}^s c_{k+1,i}D_i\right)^{\beta_{k+1}}\cdots\left(\sum_{i=1}^s c_{s,i}D_i\right)^{\beta_s}p_{\mathbf{n}}(\mathbf{w}^1)\qquad(4.2.2)$$

for $(\beta_1,...,\beta_k)\leqslant(n,...,n)\in\mathbf{Z}_+^k$ and $\beta_{k+1}+\cdots+\beta_s\leqslant r$.

Proof. If $F \in C^r(S \cup \hat{S})$, then

$$\hat{D}_{k+1}^{\beta_{k+1}} \cdots \hat{D}_s^{\beta_s} \hat{p}_{\mathbf{n}}(\mathbf{x})|_T = \left(\sum_{i=1}^{s} c_{k+1,i} D_i\right)^{\beta_{k+1}} \cdots \left(\sum_{i=1}^{s} c_{s,i} D_i\right)^{\beta_s} p_{\mathbf{n}}(\mathbf{x})\Big|_T,$$

where $\hat{D}_{k+1}^{\beta_{k+1}} \cdots \hat{D}_s^{\beta_s} \hat{p}_{\mathbf{n}}(\mathbf{x})|_T$ is a polynomial of coordinate degree $ne^1 + \cdots + ne^k$. Hence, (4.2.2) holds for $(\beta_1, ..., \beta_k) \leqslant (n, ..., n) \in \mathbf{Z}_+^k$ and $\beta_{k+1} + \cdots + \beta_s \leqslant r$.

On the other hand, suppose that (4.2.2) holds. Then

$$\hat{D}_1^{\beta_1} \cdots \hat{D}_k^{\beta_k} \hat{D}_{k+1}^{\beta_{k+1}} \cdots \hat{D}_s^{\beta_s} \hat{p}_{\mathbf{n}}(\mathbf{w}^1)$$
$$= \frac{\mathbf{n}!}{(\mathbf{n} - \beta)!} \Delta_1^{\beta_1} \cdots \Delta_k^{\beta_k} \Delta_{k+1}^{\beta_{k+1}} \cdots \Delta_s^{\beta_s} a_{(0, ..., 0)}$$

and

$$D_1^{\beta_1} \cdots D_k^{\beta_k} \left(\sum_{i=1}^{s} c_{k+1,i} D_{k+1}\right)^{\beta_{k+1}} \cdots \left(\sum_{i=1}^{s} c_{s,i} D_i\right)^{\beta_s} p_{\mathbf{n}}(\mathbf{w}^1)$$
$$= \frac{\mathbf{n}!}{(\mathbf{n} - ne^1 - \cdots - ne^k)!} \Delta_1^{\beta_{k+1}} \cdots \Delta_k^{\beta_k}$$
$$\times \sum_{|\alpha| = \beta_{k+1} + \cdots + \beta_s} b_\alpha \frac{\mathbf{n}!}{(\mathbf{n} - \alpha)!} \Delta_1^{\alpha_1} \cdots \Delta_s^{\alpha_s} R_1^{\alpha_1} \cdots R_k^{\alpha_k} a_{(0, ..., 0)}.$$

Invoking the definition of the difference operator $\Delta_1^{\beta_1} \cdots \Delta_k^{\beta_k}$ and the inversion formula in Theorem 3.1, we have (4.2.1) for $\beta_{k+1} + \cdots + \beta_s \leqslant r$. By Theorem 4.2.1, we conclude that $F \in C^r(S \cup \hat{S})$. Thus, this theorem is established.

Actually, the idea used in proving Theorem 4.2.2 can be applied to prove its generalized version, which is the following result, where the notation

$$M_{n,k,r}^s = \{(\alpha_1, ..., \alpha_s) \in \mathbf{Z}_+^s : 0 \leqslant \alpha_j \leqslant r, j = k+1, ..., s\}$$

will be used.

THEOREM 4.2.3. *Let F, S, $\hat{S}$, and T be defined as in Theorem* 4.2.2. *Let* $N_i^s \subset \Gamma_{\mathbf{n}}^s$, $i = 1, ..., 2^k$, *be lower sets such that* $R_i^{\mathbf{n}} N_i^s$, $i = 1, ..., 2^k$, *are mutually disjoint and* $\bigcup_{i=1}^{2^k} R_i^{\mathbf{n}} N_i^s = M_{n,k,r}$. *Then* $F \in C^r(S \cup \hat{S})$ *if and only if*

$$\hat{D}_1^{\beta_1} \cdots \hat{D}_k^{\beta_k} \hat{D}_{k+1}^{\beta_{k+1}} \cdots \hat{D}_s^{\beta_s} \hat{p}^{\mathbf{n}}(\mathbf{w}^i)$$
$$= D_1^{\beta_1} \cdots D_k^{\beta_k} \left(\sum_{i=1}^{s} c_{k+1,i} D_i\right)^{\beta_{k+1}} \cdots \left(\sum_{i=1}^{s} c_{si} D_i\right)^{\beta_s} p_{\mathbf{n}}(\mathbf{w}^i) \quad (4.2.3)$$

for $\beta = (\beta_1, ..., \beta_s) \in N_i^s$, $i = 1, ..., 2^k$.

EXAMPLE 4.2.1. Let $s=2$, and $S=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3, \mathbf{w}^4\rangle$ and $\hat{S}=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{u}^3, \mathbf{u}^4\rangle$ be as shown in Fig. 4.2.2, where the Bézier nets of p_3 and $\hat{p}_3$ are displayed. Write $\mathbf{u}^3-\mathbf{w}^1=c_1(\mathbf{w}^2-\mathbf{w}^1)+c_2(\mathbf{w}^3-\mathbf{w}^1)$. Then we have

$F\in C(S\cup\hat{S})$ if and only if

$$\tilde{a}_{i0}=a_{i0}, \qquad i=0, 1, 2, 3;$$

$F\in C^1(S\cup\hat{S})$ if and only if

$$\tilde{a}_{i0}=a_{i0}, \qquad i=0, 1, 2, 3,$$

and

$$\tilde{a}_{j1}=a_{j0}+c_1(j/3(a_{j0}-a_{j-1,0})+(1-j/3)(a_{j+1,0}-a_{j,0})$$
$$+c_2(a_{j1}-a_{j0}), \qquad j=0, 1, 2, 3.$$

EXAMPLE 4.2.2. Let $s=2$ and $\mathbf{n}=(5, 5)$. Furthermore, let S and $\hat{S}$ be the same as in Example 4.2.1. Define $F|_S=p_{(5,5)}$ and $F|_T=\hat{p}_{(5,5)}$. Then $F\in C^1(S\cup\hat{S})$ if and only if

$$D^{\beta_2}_{\mathbf{u}^3-\mathbf{w}^1} D^{\beta}_{\mathbf{w}^2-\mathbf{w}^1}\hat{p}_{(5,5)}(\mathbf{w}^1)$$
$$=(c_1D_{\mathbf{w}^2-\mathbf{w}^1}+c_2D_{\mathbf{w}^3-\mathbf{w}^1})^{\beta_2}(D_{\mathbf{w}^2-\mathbf{w}^1})^{\beta_1}p_{(5,5)}(\mathbf{w}^1)$$

for $0\leqslant\beta_2\leqslant 1$ and $0\leqslant\beta_1\leqslant 5$. Also, if we choose $N_1^2=N_2^2=\{(\eta_1,\eta_2): 0\leqslant\eta_1\leqslant 2, 0\leqslant\beta_2\leqslant 1\}$, then $F\in C^1(S\cup\hat{S})$ if and only if

$$D^{\beta_2}_{\mathbf{u}^3-\mathbf{w}^1} D^{\beta_1}_{\mathbf{w}^2-\mathbf{w}^1}\hat{p}_{(5,5)}(\mathbf{w}^i)$$
$$=(c_1D_{\mathbf{w}^2-\mathbf{w}^2-\mathbf{w}^1}+c_2D_{\mathbf{w}3-\mathbf{w}^1})^{\beta_2}D^{\beta_1}_{\mathbf{w}^2-\mathbf{w}^1}p_{(5,5)}(\mathbf{w}^i)$$

for $(\beta_1,\beta_2)\in N_i^2$, $i=1, 2$.

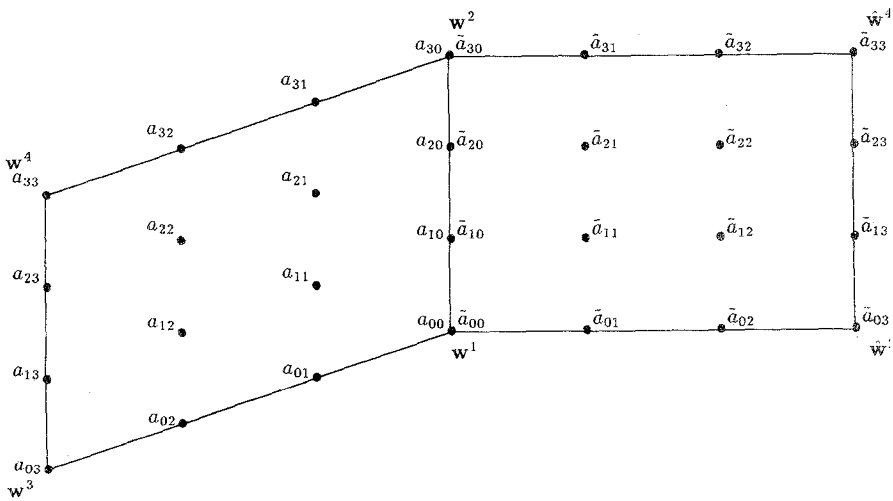

FIGURE 4.2.2

4.3. *The Mixed Partition Case* (*Triangles and Parallelograms*)

Let $S=\langle \mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2\rangle$ be a triangle and $\hat{S}=\langle \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4\rangle$ a parallelogram in $\mathbf{R}^2$. For $\mathbf{x}\in\mathbf{R}^2$, let $\lambda(\mathbf{x})=(\lambda_0(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))$, with $\lambda_0+\lambda_1+\lambda_2\equiv 1$, be the barycentric coordinate of $\mathbf{x}$ with respect to S, $\nu(\mathbf{x})=(\nu_1(\mathbf{x}), \nu_2(\mathbf{x}))$ the barycentric coordinate of $\mathbf{x}$ with respect to $\hat{S}$. Let $T=S\cap\hat{S}$. We consider only two cases: (1) $T=\{\mathbf{w}\}$, a common vertex of S and $\hat{S}$, and (2) $T=\langle \mathbf{w}^1, \mathbf{w}^2\rangle$, a common edge of S and $\hat{S}$.

Let us first study the case where $T=\langle \mathbf{w}^1, \mathbf{w}^2\rangle$ (cf. Fig. 4.3.1). Rewrite $S=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{u}^2\rangle$ and $\hat{S}=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{v}^3, \mathbf{v}^4\rangle$. Assume, without loss of generality, that

$$\begin{aligned}\lambda_1(\mathbf{w}^1)&=0=\nu_1(\mathbf{w}^1)\\ \lambda_1(\mathbf{w}^2)&=1=\nu_1(\mathbf{w}^2)\\ \lambda_2(\mathbf{x})&=0=\nu_2(\mathbf{x}), \qquad \mathbf{x}\in T.\end{aligned}$$

Also, write $\mathbf{v}^3-\mathbf{w}^1=c_1(\mathbf{w}^2-\mathbf{w}^1)+c_2(\mathbf{u}^2-\mathbf{w}^1)$. Let F be a piecewise polynomial function defined on $S\cup\hat{S}$ by

$$F|_S=p_n(\mathbf{x})=\sum_{\substack{|\beta|=n\\ \beta\in\mathbf{Z}_+^3}} a_\beta\phi_\beta^n(\lambda(\mathbf{x}))$$

and

$$F|_{\hat{S}}=\hat{p}_n(\mathbf{x})=\sum_{\alpha\leqslant(n,n)} \hat{a}_\alpha\hat{\phi}_\alpha^{(n,n)}(\nu(\mathbf{x})).$$

Furthermore, define another degree raising operator R by

$$Ra_\beta=\frac{\beta_0}{|\beta|}a_{\beta-e^0}+\frac{\beta_1}{|\beta|}a_{\beta-e^1}+\frac{\beta_2}{|\beta|}a_{\beta-e^2}.$$

We are now ready to state and prove the following result.

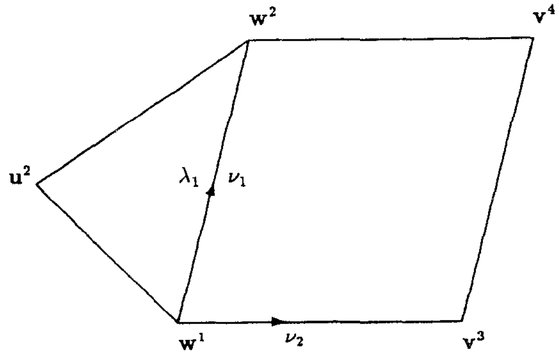

FIGURE 4.3.1

THEOREM 4.3.1. *Let* $S=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{u}^2\rangle$ *and* $\hat{S}=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{v}^3, \mathbf{v}^4\rangle$. *Then* $F\in C^r(S\cup S)$ *if and only if*

$$\Delta_2^k\, \hat{a}_{j0}=(c_1\Delta_{10}+c_2\Delta_{20})^k\, R^k a_{n-j,j,0} \qquad j=0, ..., n,\ k\leqslant r.$$

Proof. For $0\leqslant k\leqslant r$,

$$D^k_{\mathbf{v}^3-\mathbf{w}^1}\hat{p}_n(\mathbf{x})|_T=\frac{n!}{(n-k)!}\sum_{\alpha\leqslant(n,0)}\Delta_2^k\hat{a}_\alpha\hat{\phi}_\alpha^{(n,0)}(v_1(\mathbf{x}),0)$$

and

$$\begin{aligned}
&(c_1D_{\mathbf{w}^2-\mathbf{w}^1}+c_2D_{\mathbf{u}^2-\mathbf{w}^1})^k p_n(\mathbf{x})|_T\\
&=\left(\sum_{|\gamma|=k} c^\gamma\frac{k!}{\gamma!}D^{\gamma_1}_{\mathbf{w}^2-\mathbf{w}^1}D^{\gamma_2}_{\mathbf{u}^2-\mathbf{w}^1}\right)p_n(\mathbf{x})\Bigg|_T\\
&=\frac{n!}{(n-k)!}\sum_{|\gamma|=k}c^\gamma\frac{k!}{\gamma!}\sum_{|\beta|=n-k}\Delta_{10}^{\gamma_1}\Delta_{20}^{\gamma_2}a_\beta\phi_\beta^{n-k}(\lambda(\mathbf{x}))\Bigg|_T\\
&=\frac{n!}{(n-k)!}\sum_{\substack{|\beta|=n-k\\ \beta=(\beta_0,\beta_1,0)}}\sum_{|\gamma|=k}\frac{k!}{\gamma!}c^\gamma\Delta_{10}^{\gamma_1}\Delta_{20}^{\gamma_2}a_\beta\phi_\beta^{n-k}(\lambda_0(\mathbf{x}),\lambda_1(\mathbf{x}),0)\\
&=\frac{n!}{(n-k)!}\sum_{\substack{|\beta|=n\\ \beta=(\beta_0,\beta_1,0)}}\sum_{|\gamma|=k}\frac{k!}{\gamma!}c^\gamma\Delta_{10}^{\gamma_1}\Delta_{20}^{\gamma_2}R^k a_\beta\phi_\beta^{n}(\lambda_0(\mathbf{x}),\lambda_1(\mathbf{x}),0).
\end{aligned}$$

Therefore, $F\in C^r(S\cup\hat{S})$ if and only if, for $0\leqslant k\leqslant r$,

$$D^k_{\mathbf{v}^3-\mathbf{w}^1}\hat{p}_n(x)|_T=(c_1D_{\mathbf{w}^3-\mathbf{w}^1}+c_2D_{\mathbf{w}^3-\mathbf{w}^1})^k p_n(\mathbf{x})|_T$$

which gives the required result if we note that $v_1(\mathbf{x})=\lambda_1(\mathbf{x})$ for $\mathbf{x}\in T$ and $\hat{\phi}_j^{(n,0)}(v_1(\mathbf{x}),0)=\phi_{(n-j,j,0)}(1-\lambda_1(\mathbf{x}),\lambda_1(\mathbf{x}),0)$. This completes the proof of the theorem.

EXAMPLE 4.3.1. $F\in C(S\cup\hat{S})$ if and only if

$$\hat{a}_{j0}=a_{n-j,j,0}, \qquad j=0, ..., n;$$

$F\in C^1(S\cup\hat{S})$ if and only if

$$\hat{a}_{j0}=a_{n-j,j,0}, \qquad j=0, ..., n,$$

and

$$\begin{aligned}\hat{a}_{j1}&=a_{n-j,j,0}+c_1\Delta_{10}(j/na_{n-j,j-1,0}+(1-j/n)\,a_{n-j-1,j,0})\\
&\quad+c_2\Delta_{20}(j/na_{n-j,j-1,0}+(1-j/n)\,a_{n-j-1,j,0}, \qquad j=0, ..., n.\end{aligned}$$

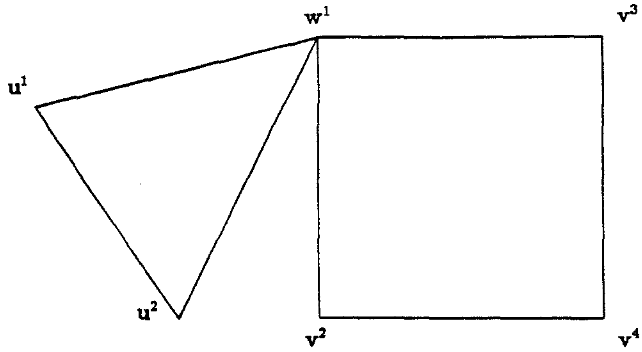

FIGURE 4.3.2

We now study the case where $T = S \cap \hat{S} = \{\mathbf{w}^1\}$ (cf. Fig. 4.3.2). Rewrite $S = \langle \mathbf{w}^1, \mathbf{u}^1, \mathbf{u}^2 \rangle$ and $\hat{S} = \langle \mathbf{w}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4 \rangle$, and let F be defined as before. Also, let

$$\mathbf{v}^2 - \mathbf{w}^1 = c_1(\mathbf{u}^1 - \mathbf{w}^1) + c_2(\mathbf{u}^2 - \mathbf{w}^1)$$

and

$$\mathbf{v}^3 - \mathbf{w}^1 = c_3(\mathbf{u}^1 - \mathbf{w}^1) + c_4(\mathbf{u}^2 - \mathbf{w}^1).$$

Then we have

THEOREM 4.3.2. *Let* $S = \langle \mathbf{w}^1, \mathbf{u}^2, \mathbf{u}^3 \rangle$ *and* $\hat{S} = \langle \mathbf{w}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4 \rangle$. *Then* $F \in C^r(S \cup \hat{S})$ *if and only if*

$$\Delta_1^{\beta_1} \Delta_2^{\beta_2} \hat{a}_{0,0} = \frac{(n-\beta_1)!\,(n-\beta_2)!}{n!(n-\beta_1-\beta_2)!} (c_1 \Delta_{10} + c_2 \Delta_{20})^{\beta_1} \times (c_4 \Delta_{10} + c_2 \Delta_{20})^{\beta_2} a_{n-\beta_1-\beta_2,0}$$

for $\beta_1 + \beta_2 \leqslant r$.

Proof. For $0 \leqslant \beta_1 + \beta_2 \leqslant r$, we have

$$D_{\mathbf{v}^2-\mathbf{w}^1}^{\beta_1} D_{\mathbf{v}^3-\mathbf{w}^1}^{\beta_2} \hat{p}_n(\mathbf{x})|_T = \frac{n!}{(n-\beta_1)!} \frac{n!}{(n-\beta_2)!} \sum_{\alpha \leqslant (n-\beta_1, n-\beta_2)} \Delta_1^{\beta_1} \Delta_2^{\beta_2} \hat{a}_\alpha \hat{\phi}_\alpha^{(n-\beta_1, n-\beta_2)}(v(\mathbf{x}))\Big|_T = \frac{n!}{(n-\beta_1)!} \frac{n!}{(n-\beta_2)!} \Delta_1^{\beta_1} \Delta_2^{\beta_2} \hat{a}_{0,0}.$$

Hence,

$$(c_1 D_{\mathbf{u}^1-\mathbf{w}^1} + c_2 D_{\mathbf{u}^2-\mathbf{w}^1})^{\beta_1} (c_3 D_{\mathbf{u}^1-\mathbf{w}^1} + c_4 D_{\mathbf{u}^2-\mathbf{w}^1})^{\beta_2} p_n(\mathbf{x})|_T$$

$$= \frac{n!}{(n-\beta_1-\beta_2)!}$$

$$\times \sum_{|\gamma|=n-\beta_1-\beta_2} (c_1 \Delta_{10} + c_2 \Delta_{20})^{\beta_1} (c_3 \Delta_{10} + c_4 \Delta_{20})^{\beta_2} a_\gamma \phi_\gamma^{n-\beta_1-\beta_2}(\hat{\lambda}(\mathbf{x}))\Big|_T$$

$$= \frac{n!}{(n-\beta_1-\beta_2)!} (c_1 \Delta_{10} + c_2 \Delta_{20})^{\beta_1} (c_3 \Delta_{10} + c_4 \Delta_{20})^{\beta_2} a_{n-\beta_1-\beta_2,0,0}.$$

It follows that $F \in C^r(S \cup \hat{S})$ if and only if

$$D^{\beta_1}_{\mathbf{v}^2-\mathbf{w}^1} D^{\beta_2}_{\mathbf{v}^3-\mathbf{w}^1} \hat{p}_n(\mathbf{x})|_T$$
$$= (c_1 D_{\mathbf{u}^1-\mathbf{w}^1} + c_2 D_{\mathbf{u}^2-\mathbf{w}^1})^{\beta_1} (c_3 D_{\mathbf{u}^1-\mathbf{w}^1} + c_4 D_{\mathbf{u}^2-\mathbf{w}^1})^{\beta_2} p_n(\mathbf{x})|_T$$

which completes the proof of the theorem.

EXAMPLE 4.3.2. $F \in C(S \cup \hat{S})$ if and only if

$$\hat{a}_{n,0,0} = a_{0,0};$$

$F \in C^1(S \cup \hat{S})$ if and only if

$$\hat{a}_{0,0} = a_{n,0,0},$$

$$\hat{a}_{1,0} = (1 - c_1 - c_2)\, a_{n,0,0} + c_1 a_{n-1,1,0} + c_2 a_{n-1,0,1},$$

and

$$\hat{a}_{0,1} = (1 - c_3 - c_4)\, a_{n,0,0} + c_3 a_{n-1,1,0} + c_4 a_{n-1,0,1}.$$

5. VERTEX SPLINES AND SUPER SPLINE SPACES

In this section, we are going to construct vertex splines on a given simplicial or parallelepiped partitioned region in $\mathbf{R}^s$, where $s \geq 2$, and a mixed partitioned region consisting of triangles and parallelograms in $\mathbf{R}^2$ by using the results obtained in the previous sections. Before going into the details, let us first describe these regions and give a general definition of vertex splines and introduce the notion of the related super spline spaces that will be applied in the next section for L^2 and l^2 approximations with interpolatory constraints.

A simplex with $k+1$ vertices in $\mathbf{R}^s$ and positive k-dimensional volume is called a k-simplex, $0 \leq k \leq s$, and a point will be called a 0-simplex for consistency. For any s-simplex $S = \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle$, each k-simplex $\langle \mathbf{v}^{i_0}, ..., \mathbf{v}^{i_k} \rangle$, where $0 \leqslant i_0 < \cdots \leqslant i_k \leqslant s$, is called a k-facet of S if $\langle \mathbf{v}^{i_0}, ..., \mathbf{v}^{i_k} \rangle \subseteq \partial S$, the $(s-1)$ boundary of S.

A parallelepiped in $\mathbf{R}^s$ with positive k-simensional volume is called a k-parallelepiped, $0 \leqslant k \leqslant s$. Similarly, a point will also be called a 0-parallelepiped. For an s-parallelepiped $S = \langle \mathbf{w}^1, ..., \mathbf{w}^{2^s} \rangle \subset \mathbf{R}^s$, an $(s-1)$-parallelepiped $\langle \mathbf{w}^{i_1}, ..., \mathbf{w}^{i_{2^{s-1}}} \rangle$, where $1 \leqslant i_1 < \cdots < i_{2^{s-1}} \leqslant 2^s$, is called an $(s-1)$-facet of S if it is a subset of ∂S. For $k = s-2, ..., 0$, inductively, a k-parallelepiped $\langle \mathbf{w}^{j_1}, ..., \mathbf{w}^{j_{2^k}} \rangle$, $1 \leqslant j_1 < \cdots < j_{2^k} \leqslant 2^s$, is called a k-facet of S if it is a subset of the boundary of some $(k+1)$-facet of S.

DEFINITION 1. A region $D \subset \mathbf{R}^s$ which is the union of a finite number of s-simplices (or s-parallelepipeds, respectively) $S_1, ..., S_N$ is called a simplicial (or parallelepiped, respectively) partitioned region if it satisfies

(i) $\operatorname{int}(S_i) \cap \operatorname{int}(S_j) = \varnothing$, $i \neq j$; and

(ii) either $S_i \cap S_j = \varnothing$ or $S_i \cap S_j$ is a k-simplex (or k-paralelepiped, respectively) which is a common k-facet of S_i and S_j for some k, $0 \leqslant k \leqslant s-1$.

DEFINITION 2. A mixed partitioned region $D \subset \mathbf{R}^2$ is the union of a finite number of triangles and 2-parallelepipeds (parallelograms) $S_1, ..., S_N$ which satisfies

(i′) $\operatorname{int}(S_i) \cap \operatorname{int}(S_j) \varnothing$, $i \neq j$; and

(ii′) either $S_i \cap S_j = \varnothing$ or $S_i \cap S_j$ is a point which is a common vertex of S_i and S_j or $S_i \cap S_j$ is a common edge of S_i and S_j.

In this paper, we will not study vertex splines on a mixed partitioned region in $\mathbf{R}^s$, $s > 2$, which contains other convex hulls such as prism.

Let $D \subset \mathbf{R}^s$ be a region considered in Definitions 1 or 2 above. For r, $d \in \mathbf{Z}_+$ with $0 \leqslant r < d$, let

$$S_d^r = S_d^r(D) = \{ f \in C^r(D) : f|_{S_i} \in \pi_d^s(S_i), i = 1, ..., N \}$$

be the multivariate spline space of degree d and order r of smoothness on D, where if S_i is an s-simplex, $\pi_d^s(S_i)$ is the polynomial space of total degree d, and if S_i is an s-parallelepiped, $\pi_d^s(S_i) = \tilde{\pi}_{\mathbf{d}}^s(S_i)$ is the polynomial space of coordinate degree $\mathbf{d} = (d, ..., d)$.

DEFINITION 3. Let

$$\hat{S}_d^r = \hat{S}_d^r(D) = \{ f \in S_d^r(D) : f \in C^{2^{s-j-1}r} \text{ across each } j\text{-dimensional manifold of the partition of } D, 0 \leqslant j < s \}.$$

$\hat{S}_d^r$ will be called the space of super splines.

Remark 1. For $s=1$, $\hat{S}^r_d = S^r_d$.
We are now ready to define vertex splines.

DEFINITION 4. Let $0 \leqslant k \leqslant s$. A super spline $f \in V^s_k \subset \hat{S}^r_d(D)$ is called a k-vertex spline if there exists a k-simplex or k-parallelepiped K such that the support of f is the union of all cells (simplices or parallelepipeds) in D with K as their common k-facet and that f or one of its first or higher order partial dervatives is nontrivial on K. The union of all V^s_k, $k=0, ..., s$ is the collection of all vertex splines in $\hat{S}^r_d(D)$.

Remark 2. The notion of vertex splines was first introduced in [11], where only bivariate 0-vertex splines were studied. We will see that vertex splines always exist if we assume $d \geqslant 2^s r+1$. In general, a vertex spline with degree $d \leqslant 2^s r$ may also be constructed on a simplicial region D with some restriction on the geometry. See [11] for $s=2$, $r=1$, and $d=4$, and [10] for $s=2$ and arbitrary d and r.

Remark 3. For $d \geqslant 2^s r+1$, an element in $\hat{S}^r_d$ restricted to each s-simplex of D can also be considered as a Hermite element with directional derivatives at the vertices instead of normal derivatives at points inside the k-facets of simplex $0<k<s$. See [22, 24] for references on Hermite elements in $\mathbf{R}^s$. Furthermore, adopting the notion of vertex splines instead of finite elements, we may consider finite element analysis from the viewpoint of approximation theory. We hope that vertex splines will then play an important role in cross-fertilizing the two important fields of approximation theory and finite element analysis.

5.1. *Simplicial Partitioned Regions*

Let us first establish the following theorem on the existence of vertex splines on any given simplicial partitioned region by outlining the construction procedure.

THEOREM 5.1.1. *Let $d \geqslant 2^s r+1$, $r \geqslant 0$, and let D be a given simplicial partitioned region. For each k-simplex T_k in D, $0 \leqslant k \leqslant s$, there exists at least one vertex spline $f \in V^s_k \subset \hat{S}^r_d$ supported on the union of those s-simplices of D that share T_k as the common intersection, with only one exceptional case: there is no nontrivial 2-vertex spline in $S^1_5(D)$, where $D \subset \mathbf{R}^2$.*

Proof. We start with the simple case where $s=2$. For completeness, we include the construction procedure of 0-vertex splines studied in [11].

(i) *Construction of $V^2_0 \subset \hat{S}^r_d$, $D \subset \mathbf{R}^2$.*

Let $\mathbf{v}^1$ be a vertex (or 0-simplex) of D and $S_\nu = \langle \mathbf{v}^1, \mathbf{v}^{1,\nu}, \mathbf{v}^{2,\nu} \rangle$, $\nu=1, ..., l$, be all the triangles (or 2-simplices) of D which share $\mathbf{v}^1$ as the common vertex. Without loss of generality, we assume that S_ν and $S_{\nu+1}$ share an edge $\langle \mathbf{v}^1, \mathbf{v}^{2,\nu}\rangle$ (or 1-simplex) as their intersection, where $\mathbf{v}^{2,\nu} = \mathbf{v}^{1,\nu+1}$ (and $S_{l+1} := S_1$ if $\mathbf{v}^1$ is an interior vertex). Let F be a piecewise polynomial function supported on $\bigcup_{\nu=1}^{l} S_\nu$ and defined by

$$F\,|_{S_\nu} = \sum_{|\alpha| = d} a_\alpha^\nu \phi_\alpha^d, \qquad \nu = 1, ..., l.$$

To determine $F \in \hat{S}_d^r$, we specify its Bézier nets a_α^ν as follows:

(a) We require that

$$D^\beta F(\mathbf{v}^1) = c_\beta, \qquad |\beta| \leqslant 2r \tag{5.1.1}$$

and

$$D^\beta F(\mathbf{v}^{1,\nu}) = 0 = D^\beta F(\mathbf{v}^{2,\nu}), \qquad |\beta| \leqslant 2r, \tag{5.1.2}$$

where $\{c_\beta : |\beta| \leqslant 2r\}$ is a parameter set of real numbers which are not all zeros.

Let $N_j^0 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 + \alpha_3 = d,\, d - 2r \leqslant \alpha_j \leqslant d\}, j = 1, 2, 3$. Then it is clear that the requirements (5.1.1) and (5.1.2) uniquely determine the Bézier coefficients a_α^ν, for $\alpha \in N_1^0 \cup N_2^0 \cup N_3^0$, $\nu = 1, ..., l$, by the application of Theorem 3.1.2.

(b) For $F|_{S_\nu}$, we require that

$$D^{\beta_1}_{\mathbf{v}^1 - \mathbf{v}^{1,\nu}} D^{\beta_2}_{\mathbf{v}^{2,\nu} - \mathbf{v}^{1,\nu}} F(\mathbf{v}^{1,\nu}) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^1, \tag{5.1.3}$$

where $\hat{N}_1 = \{(\beta_1, \beta_2) : 2r < \beta_1 + \beta_2,\, \beta_1 \leqslant r,\, \beta_2 \leqslant d - 2r - 1\}$. We also require that

$$D^{\beta_1}_{\mathbf{v}^{1,\nu} - \mathbf{v}^{2,\nu}} D^{\beta_2}_{\mathbf{v}^1 - \mathbf{v}^{2,\nu}} F(\mathbf{v}^{2,\nu}) = 0, \qquad (\beta_1, \beta_2) \in N^1. \tag{5.1.4}$$

Hence, by Theorem 3.1.3 the requirements (5.1.4) and (5.1.4) uniquely determine the corresponding coefficients a_α^ν. Now we obtain

$$D^{\beta_1}_{\mathbf{v}^{2,\nu+1} - \mathbf{v}^{1,\nu+1}} D^{\beta_2}_{\mathbf{v}^1 - \mathbf{v}^{1,\nu+1}} F(\mathbf{v}^{1,\nu+1}), \qquad (\beta_1, \beta_2) \in \hat{N}^1,$$

from some of the a_α^ν which have already been determined and we determine the corresponding Bézier nets $a_\alpha^{\nu+1}$ by applying Theorem 4.1.2. Then the coefficients a_α^ν, $\alpha \in \bigcup_{i=1}^3 N_i^1$, where

$$N_i^1 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 + \alpha_3 = d,\, 0 \leqslant \alpha_i \leqslant r\} \Big\backslash \bigcup_{k=1}^{3} N_k^0,$$

are uniquely determined by the requirements (5.1.3) and (5.1.4).

(c) For $F|_{S_\nu}$, we require that

$$D^{\beta_1}_{\mathbf{v}^{1,\nu} - \mathbf{v}^{2,\nu}} D^{\beta_2}_{\mathbf{v}^{2,\nu} - \mathbf{v}^1} F(\mathbf{v}^1) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^2, \tag{5.1.5}$$

where $\hat{N}^2 = \{(\beta_1, \beta_2), \beta_1, \beta_2 \geq r+1, \beta_1 + \beta_2 \leq d-r-1\}$. This is equivalent to determining the a_α^ν with α in

$$N^2 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 + \alpha_3 = d\} \Big\backslash \bigcup_{j=1}^{3} (N_j^1 \cup N_j^0).$$

Thus, we note that the requirements (5.1.1)–(5.1.5) have uniquely determined a polynomial of total degree d on each 2-simplex S_ν, $\nu = 1, ..., l$, for the given data c_β, $|\beta| \leq 2r$, by the use of Theorem 3.1.5. That is, F is completely determined. Clearly, $F \in C^r(D)$ by Theorem 4.1.3 and $F \in C^{2r}$ at $\mathbf{v}^1$, $\mathbf{v}^{1,\nu}$, and $\mathbf{v}^{2,\nu}$, $\nu = 1, ..., l$, so that $F \in C^{2r}$ at all the 0-simplices of D since F is only supported on the union of these simplices. Hence, F is a vertex spline in $V_0^2 \subset \hat{S}_d^r \subset S_d^r$.

(ii) *Construction of* $V_1^2 \subset \hat{S}_d^r$, $D \subset \mathbf{R}^2$.

Let $\langle \mathbf{v}^0, \mathbf{v}^1 \rangle$ be an edge (or 1-simplex of D) and $S_1 = \langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2 \rangle$ and $S_2 = \langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^3 \rangle$ be two triangles (or 2-simplices) sharing $\langle \mathbf{v}^0, \mathbf{v}^1 \rangle$ as the common edge. Suppose that F is a piecewise polynomial supported on $S_1 \cup S_2$ defined by

$$F|_{S_i} = \sum_{|\alpha| = d} a_\alpha^i \phi_\alpha^d, \qquad i = 1, 2.$$

To determine $F \in \hat{S}_d^r$, we specify its Bézier nets a_α^1 and a_α^2 as follows:

(a) We require that

$$D^\beta F(\mathbf{v}^i) = 0, \qquad |\beta| \leq 2r,\ i = 0, 1, 2, 3. \tag{5.1.6}$$

By Theorem 3.1.2, we know that the requirement (5.1.6) uniquely determines a_α^i, $\alpha \in N_j^0$, $j = 1, 2, 3$, and $i = 1, 2$.

(b) For $F|_{S_1}$, we require that

$$D_{\mathbf{v}^2 - \mathbf{v}^0}^{\beta_1} D_{\mathbf{v}^1 - \mathbf{v}^0}^{\beta_2} F(\mathbf{v}^0) = c_{\beta_1, \beta_2}, \qquad (\beta_1, \beta_2) \in \hat{N}^1, \tag{5.1.7}$$

where c_{β_1, β_2} are constants which are not all equal to zero. We compute

$$D_{\mathbf{v}^3 - \mathbf{v}^0}^{\beta_1} D_{\mathbf{v}^1 - \mathbf{v}^0}^{\beta_2} F(\mathbf{v}^0), \qquad (\beta_1, \beta_2) \in \hat{N}^1,$$

from the corresponding coefficients a_α^1 which have been determined by (5.1.6) and (5.1.7), and by applying Theorem 4.1.2 we may use these derivative values to determine the corresponding a_α^2. We also require that

$$\begin{aligned} &D_{\mathbf{v}^0 - \mathbf{v}^i}^{\beta_1} D_{\mathbf{v}^j - \mathbf{v}^i}^{\beta_2} F(\mathbf{v}^i) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^1,\ i = 1, 2, \text{ and} \\ &D_{\mathbf{v}^1 - \mathbf{v}^i}^{\beta_1} D_{\mathbf{v}^0 - \mathbf{v}^i}^{\beta_2} F(\mathbf{v}^i) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^1,\ i = 1, 2. \end{aligned} \tag{5.1.8}$$

Then the coefficients a^i_α, $\alpha \in N^1_j$, $j=1, 2, 3$, $i=1, 2$, are uniquely determined by the requirements in (5.1.7) and (5.1.8) along the line of Theorem 3.1.3.

(c) For $F|_{S_i}$, $i=1, 2$, we require that

$$D^{\beta_1}_{\mathbf{v}^{i+1}-\mathbf{v}^0} D^{\beta_2}_{\mathbf{v}^i-\mathbf{v}^0} F(\mathbf{v}^0)=0, \qquad (\beta_1, \beta_2) \in \hat{N}^2. \tag{5.1.9}$$

Clearly, we can see that the requirements in (5.1.6)–(5.1.9) uniquely determine the polynomials $F|_{S_1}$ and $F|_{S_2}$ by the application of Theorem 3.1.5 for the given data $\{c_{\beta_1,\beta_2}: (\beta_1, \beta_2) \in \hat{N}^1\}$. Hence,$F$ is completely determined. Moreover, $F \in C^r(D)$, by Theorem 4.1.3, and $F \in C^{2r}$ at $\mathbf{v}^i$, $i=0, 1, 2, 3$, because of the requirements in (5.1.6). Thus, $F \in C^{2r}$ at all the 0-simplices of D. Therefore, F is a vertex spline in $V^2_1 \subset \hat{S}^r_d \subset S^r_d$.

(iii) *Construction of* $V^2_2 \subset \hat{S}^r_d$, $D \subset \mathbf{R}^2$.

Let $d>5$ if $r=1$ and $d \geqslant 4r+1$ if $r>1$. Let $\langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2 \rangle$ be a triangle (or 2-simplex) in D and F a piecewise polynomial function with support $\langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2 \rangle$. Write $F=\sum_{|\alpha|=d} a_\alpha \phi^d_\alpha$. To determine $F \in \hat{S}^r_d$, we specify a_α as follows:

(a) We require that

$$D^\beta F(v^i)=0, \qquad |\beta| \leqslant 2r, i=0, 1, 2. \tag{5.1.10}$$

(b) We require that

$$\begin{aligned} &D^{\beta_1}_{\mathbf{v}^1-\mathbf{v}^0} D^{\beta_2}_{\mathbf{v}^2-\mathbf{v}^0} F(\mathbf{v}^0)=0, \\ &D^{\beta_1}_{\mathbf{v}^0-\mathbf{v}^1} D^{\beta_2}_{\mathbf{v}^2-\mathbf{v}^1} F(\mathbf{v}^0)=0, \qquad (\beta_1, \beta_2) \in \hat{N}^1. \\ &D^{\beta_1}_{\mathbf{v}^1-\mathbf{v}^2} D^{\beta_2}_{\mathbf{v}^0-\mathbf{v}^2} F(\mathbf{v}^0)=0, \end{aligned} \tag{5.1.11}$$

(c) We require that

$$D^{\beta_1}_{\mathbf{v}^1-\mathbf{v}^0} D^{\beta_2}_{\mathbf{v}^2-\mathbf{v}^0} F(\mathbf{v}^0)=c_{\beta_1,\beta_2}, \qquad (\beta_1, \beta_2) \in \hat{N}^2. \tag{5.1.12}$$

By Theorem 3.1.5, the polynomial F on $\langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2 \rangle$ is uniquely determined by the requirements (5.1.10)–(5.1.12). That $F \in C^{2r}$ at $\mathbf{v}^i$, $i=0, 1, 2$, is clear from (5.1.10), and $F \in C^r(D)$ from Theorem 4.1.3. Hence, F is in $V^2_2 \subset \hat{S}^r_d \subset S^r_d$.

The procedure in constructing bivariate vertex splines in V^2_0, V^2_1, V^2_2 can easily generalized to the higher-dimensional setting. Let us describe the general procedure for constructing vertex splines in $V^s_k \subset \hat{S}^r_d(D)$, $D \subset \mathbf{R}^s$, $s>2$, $0 \leqslant k \leqslant s$:

Fix a k-simplex T^s_k of a given simplicial partitioned domain D and let $S_1, ..., S_l$ be all those s-simplices of D which share T^s_k as their common k-facet. Write $S_\nu = \langle \mathbf{v}^{\nu,0}, ..., \mathbf{v}^{\nu,s} \rangle$, $\nu=1, ..., l$. Denote by T_{ji}, $i=1, ..., l_j$ the

j-simplices of $\bigcup_{v=1}^{l} S_v$, $j=0, ..., s-1$. Let F be a piecewise polynomial function of total degree $d \geq 2^s r+1$ supported on $\bigcup_{v=1}^{l} S_v$. Write

$$F|_{S_v} = \sum_{|\alpha|=d} a_\alpha^v \phi_{v,\alpha}^n, \qquad v=1, ..., l.$$

In order to have $F \in \hat{S}_d^r$, we specify its Bézier net a_α^v as follows:

(a) For $j=0$ and each T_{0i}, $i=1, ..., l_0$, we require that

$$D^\beta F(T_{0i}) = \begin{cases} 0 & \text{if } T_{0i} \neq T_k^s \\ c_\beta, & \text{if } T_{0i} = T_k^s \end{cases} \quad \text{for } |\beta| \leq 2^{s-1} r, \tag{5.1.13}$$

where $\{c_\beta : |\beta| \leq 2^{s-1} r\}$ is a parameter set which contains at least one nonzero element. Let $N_{0_j} = \{\alpha \in \mathbf{Z}^{s+1} : |\alpha| = d,\ \alpha_0 + \cdots + \alpha_{j-1} + \alpha_{j+1} + \cdots + \alpha_s \leq 2^{s-1} r\}$, $j=0, ..., s$.

(b) For $j=1, ..., s-1$ and each T_{ji}, $i=1, ..., l_j$, we let S_m, $m \in \{n_{ji,1}, ..., n_{ji,l(ji)}\}$ be those s-simplices of S_v, $v=1, ..., l$, which share T_{ji} as their common j-facet. Since there are $\binom{s+1}{j+1}$ choices of $j+1$ indices $\{u_0, ..., u_j\}$ from the index set $\{0, ..., s\}$, we may enumerate the $\binom{s+1}{j+1}$ choices by any ordering, and for each u, $1 \leq u \leq \binom{s+1}{j+1}$, let

$$N_{j,u} = \{\alpha \in \mathbf{Z}_+^{s+1} : |\alpha| = d, \alpha_{u_{j+1}} + \cdots + \alpha_{u_s} \leq 2^{s-j-1} r\},$$

where $\{u_{j+1}, ..., u_s\} = \{0, ..., s\} \setminus \{u_0, ..., u_j\}$. Now, for a given s-simplex S_m, write $T_{ji} = \langle \mathbf{v}^{m,u_0}, ..., \mathbf{v}^{m,u_j} \rangle$, $m = n_{ji,1}$, with a fixed vertex $\mathbf{v}^{m,u_0}$. We require that

$$D_0^\beta F(\mathbf{v}^{m,u_0}) = \begin{cases} 0 & \text{if } T_{ji} \neq T_k^s \\ c_\beta, & \text{if } T_{ji} = T_k^s \end{cases} \tag{5.1.14}$$

for $\beta \in c_{u_0} N_{ji}$, where

$$D_0^\beta = \prod_{k=1}^{s} (D_{\mathbf{v}^{m,u_k} - \mathbf{v}^{m,u_0}})^{\beta_k},$$

$$N_{ji} = N_{j,u} \Big\backslash \left(\bigcup_{t=0}^{j-1} \bigcup_{1 \leq u \leq \binom{s+1}{t+1}} N_{t,u} \right),$$

and $\{c_\beta : \beta \in c_{u_0} N_{ji}\}$ is a parameter set which contains at least one nonzero element. For the other simplices S_k, $k \in \{n_{ji,2}, ..., n_{ji,l(ji)}\}$, we compute $\hat{D}^\beta F(\mathbf{v}^{m,u_0})$ from $F|_{S_m}$ and then use these interpolating data to determine the corresponding coefficients of $F|_{S_k}$ by applying Theorem 4.1.2.

(c) For $j=s$ and each S_v, $v=1, ..., l$, we require that

$$D_0^{\beta} F(\mathbf{v}^{v,0}) = \begin{cases} 0 & \text{if } S_v \neq T_k^s \\ c_\beta & \text{if } S_v = T_k^s \end{cases} \tag{5.1.15}$$

for $\beta \in c_0 N_s$, where

$$N_s = \{\alpha \in \mathbf{Z}^{s+1} : |\alpha| = n\} \Big\backslash \left(\bigcup_{t=0}^{s-1} \bigcup_{1 \leqslant u \leqslant \binom{s+1}{t+1}} N_{t,u} \right)$$

and $\{c_\beta : \beta \in c_0 N_s\}$ is a set of real numbers containing at least one nonzero number.

By applying Theorem 3.1.5, we can see that F is uniquely determined on each S_v, $v=1, ..., l$, since it is easy to verify that $\{N_{t,u} : 1 \leqslant u \leqslant \binom{s+1}{t+1}, 0 \leqslant t \leqslant s-1\}$ can be arranged as lower sets attached to the vertices $\mathbf{v}^{v,0}, ..., \mathbf{v}^{v,s}$. Also, that $F \in \hat{S}_d^r$ is guaranteed by the requirements (5.1.13)–(5.1.15) and Theorem 4.1.3. This establishes the theorem.

1° For a simplicial partition region $D \subset \mathbf{R}^s$, $r \geqslant 0$ and $d \geqslant 2^s r + 1$, we construct basic vertex splines which constitute a basis of the super spline space $\hat{S}_d^r$ as follows.

2° For each 0-simplex $T_{0,i}$, $i=1, ..., l_0$, of D and for each $\gamma \in \mathbf{Z}_+^s$ with $|\gamma| \leqslant 2^{s-1} r$, we let $V_{0,i}^{\gamma} \in V_0^s \subset \hat{S}_d^r$ with support given by the union of those s-simplices that share $T_{0,i}$ as their common 0-facet with parameters $c_\beta = \delta_{\gamma\beta}$, $|\beta| \leqslant 2^{s-1} r$, where $\delta_{\gamma\beta}$ is the usual Kronecker delta; that is, $\delta_{\gamma,\beta} = 0$ for $\beta \neq \gamma$ and $=1$ for $\beta = \gamma$.

3° For each j-simplex T_{ji}, $i=1, ..., l_j$, of D and for each $\gamma \in N_{ji1}$, let V_{ji}^{γ} be an element of $V_j^s \subset \hat{S}_d^r$ with support given by the union of all those s-simplices of D that share T_{ji} as their common j-facet, and with parameters $c_\beta = \delta_{\gamma\beta}$, $\beta \in \hat{N}_{ji}$, $j=1, ..., s-1$, $i=1, ..., l_j$, where $\hat{N}_{ji} = c_{u_0(ji)} N_{ji}$.

For each S_v, $v=1, ..., l$ of D and $\gamma \in N_s$, let $V_{s,v}^{\gamma} \in V_s^s \subset \hat{S}_d^r$ with S_v as its support and parameters $c_\beta = \delta_{\gamma\beta}$, $\beta \in N_s$.

Let B be the collection of all vertex splines so constructed. Clearly, B is a linearly independent set of functions in $\hat{S}_d^r$. In fact, we have

THEOREM 5.1.2. *B is a basis of $\hat{S}_d^r$.*

Proof. We need to prove only that B spans $\hat{S}_d^r$. For each $f \in \hat{S}_d^r$, we claim that f is a linear combination of elements in B. Indeed, let $f_1 = f - \sum_{i=1}^{l_0} \sum_{|\gamma| \leqslant 2^{s-1} r} D^{\gamma} f(\mathbf{v}^i) V_{0,i}^{\gamma}$. Then $f_1 \in \hat{S}_d^r$ and satisfies $D^{\gamma} f_1(\mathbf{v}^i) = 0$, for $|\gamma| \leqslant 2^{s-1} r$, $i=1, ..., l_0$. Also, let $f_2 = f_1 - \sum_{i=1}^{l_1} \sum_{\gamma \in \hat{N}_{1i}} D_0^{\gamma} f(\mathbf{v}^{n_{1i,u_0(1i)}}) V_{1i}^{\gamma}$. Then $f_2 \in \hat{S}_d^r$ and satisfies $D^{\gamma} f_2(\mathbf{v}^i) = 0$, for $|\gamma| \leqslant 2^{s-1} r$, $i=1, ..., l_0$, as well as

$D_0^\gamma f_2(\mathbf{v}^{n_{1i},u_0(1i)})=0$, for $\gamma\in\hat{N}_{1i}$, $i=1,...,l_1$. We repeat this procedure until we have an f_s in $\hat{S}_d^r$ that differs from f by a linear combination of elements in B and that satisfies $D^\gamma f_s(\mathbf{v}^i)=0$, for $|\gamma|\leqslant 2^{s-1}r$, $i=1,...,l_0$; $D^\gamma f(\mathbf{v}^{N_{ji},k})=0$, for $\gamma\in\hat{N}_{ji}$, $i=1,...,l_j$, $j=1,...,s-1$; and $D_0^\gamma f(\mathbf{v}^{v,u_0(ji)})=0$, for N_s, $v=1,...,l$. On the other hand, for each $v=1,...,l$, $f_s(\mathbf{x})|_{S_v}$ is a polynomial of degree d satisfying these zero interpolation conditions. By Theorem 3.1.5, $f_s|_{S_v}=0$ for $v=1,...,l$. Hence, $f_s\equiv 0$ and this completes the proof of the theorem.

Moreover, we have the following result concerning how well the super spline subspace $\hat{S}_d^r$ approximates.

THEOREM 5.1.3. *Suppose that* $f\in C^{d+1}(D)$ *with* $d\geqslant 2^s r+1$. *Then*

$$\inf_{s\in\hat{S}_d^r(D)}\|f-s\|_\infty\leqslant Ch^{d+1}\max_{|\beta|=d-1}\|D^\beta f\|_\infty$$

for some constant C independent of h and f, where h is the maximum of the diameters of the simplices S_v, $v=1,...,l$.

Proof. Let $M: C^{d+1}(D)\to\hat{S}_d^r$ be defined by

$$\begin{aligned}Mf(\mathbf{x})=&\sum_{i=1}^{l_0}\sum_{|\gamma|\leqslant 2^{s-1}r}D^\gamma f(\mathbf{v}^i)\,V_{0i}^\gamma(\mathbf{x})\\&+\sum_{j=1}^{s-1}\sum_{i=1}^{l_j}\sum_{\gamma\in\hat{N}_{ji}}D_0^\gamma f(\mathbf{v}^{n_{ji},u_0(ji)})\,V_{ji}^\gamma(\mathbf{x})\\&+\sum_{i=1}^{l}\sum_{\gamma\in N_s}D_0^\gamma f(\mathbf{v}^{j0})\,V_{sj}^\gamma(\mathbf{x})\end{aligned}$$

for any $f\in C^{d+1}(D)$. Clearly, M is an interpolation operator and by induction on the number of s-simplices in D and recalling Theorem 3.1.3, we can prove that $Mp=p$ for all $p\in\pi_d$, where π_d is the space of all polynomials of total degree $\leqslant d$. Hence, for any fixed x in D,

$$F(f)=f(\mathbf{x})-Mf(\mathbf{x})$$

defines a linear functional F on $C^{d+1}(D)$ which clearly satisfies the following two properties:

(a) $|F(f)|\leqslant C_1\sum_{i=0}^d h^k\|f\|_k$, where $\|f\|_k$, where $\|f\|_k=\max_{|\beta|=k}\|D^\beta f\|_\infty$ and C_1 is a constant independent of f and d, and

(b) $F(p)=0$ for all $p\in\pi_d$.

By a result along the line of Bramble and Hilbert [6] or the proof of Theorem 5.2.3 to follow, we have

$$|F(f)|\leqslant Ch^{d+1}\|f\|_{d+1},$$

where C is a constant independent of f, d, and $\mathbf{x}$. That is,

$$|f(\mathbf{x}) - Mf(\mathbf{x})| \leqslant Ch^{d+1} \|f\|_{d+1}, \qquad \mathbf{x} \in D,$$

which establishes the theorem.

Several remarks are in order.

Remark 5.1.1. For $s=2$, a different formulation of Theorem 5.1.3 is known in the finite element literature (cf. [29, 30, 21]).

Remark 5.1.2. Though $\hat{S}^r_d = S^r_d$ when $s=1$, $\hat{S}^r_d$ is a proper subspace of S^r_d for $s \geqslant 2$. For $s=2$ and $d \geqslant 4r+1$, we can even compare the dimensions of $\hat{S}^r_d$ and S^r_d. For a simplicial partitioned domain $D \subset \mathbf{R}^2$, let

$V =$ number of vertices (0-simplices) of D,

$E =$ number of edges (1-simplices) of D,

$T =$ number of triangles (2-simplices) of D.

We have the following result on the dimension of the space of super splines.

THEOREM 5.1.4. *Let $r \geqslant 0$ and $d \geqslant 4r+1$. Then*

$$\dim \hat{S}^r_d = (r+1)(2r+1)\, V + \left((r+1)(d-4r-1) + \frac{r(r+1)}{2} \right) E$$
$$\frac{(d-3r-2)(d-3r-1)}{2}\, T.$$

Remark 5.1.3. In a recent paper by Alfeld and Schumaker [1], the dimension of the spline space $S^r_d(D)$ whee $d \geqslant 4r+1$ was determined to be

$$\dim S^r_d(D) = \frac{(d+1)(d+2)}{2} + \frac{(d-r)(d-r+1)}{2}$$
$$\times E_I - \frac{d^2+3d-r^2-3r}{2} V_I + \sigma(r),$$

where E_I and V_I denote, respectively, the number of interior vertices and the number of interior edges, and

$$\sigma(r) = \sum_{i=1}^{V_I} \sum_{j=1}^{d-r} (r+j+1-je_i)_+$$

with e_i denoting the number of edges of different slopes meeting at the ith interior vertex.

Let $V_B = V - V_I$ be the number of boundary vertices. It is clear that $V_B \geqslant 3$. Then by using the well-known formulas

$$E_I = 3V_I + V_B - 3, \qquad T = 2V_I + V_B - 2,$$

it is not difficult to arrive at the following result.

COROLLARY 5.1. *Let $r \geqslant 0$ and $d \geqslant 4r+1$. Then*

$$\dim S_d^r - \dim \hat{S}_d^r = \tfrac{3}{2}r(r+1)\, V_I + r(r+1)(V_B - 3) + \sigma(r).$$

Hence, for $r > 0$, $\hat{S}_d^r$ is a proper subspace of S_d^r unless the partitioned region D consists of a single triangle.

Proof of Theorem 5.1.4. By Theorem 5.1.3, since

$$\begin{aligned}\hat{S}_d^r = \operatorname{span}\{ & V_{0i}^\gamma : i = 1, ..., V, |\gamma| \leqslant 2r\} \\ & \cup \{V_{1i}^\gamma : i = 1, ..., E, \gamma \in N_1\} \cup \{V_{2i}^\gamma : i = 1, ..., T, \gamma \in N_2\},\end{aligned}$$

where

$$N_1 = \{\gamma \in \mathbf{Z}_+^2 : 2r < \gamma_1 + \gamma_2, 0 \leqslant \gamma_1 \leqslant r, 0 \leqslant \gamma_2 \leqslant d - 2r - 1\}$$

and

$$N_2 = \{\gamma \in \mathbf{Z}_+^2 : 2r < \gamma_1 + \gamma_2 \leqslant d - r - 1, r < \gamma_1 < d - 2r, r < \gamma_2 < d - 2r\},$$

it follows that the cardinality of N_1 is $r(r+1)/2 + (r+1)(d-4r-1)$ and the cardinality of N_2 is $(d-3r-2)(d-3r-1)/2$. This completes the proof of the theorem.

5.2. *Parallelepiped Partitioned Region*

We first prove the following existence theorem of vertex splines on a given parallelepiped partitioned region for $d \geqslant 2^s r + 1$ by outlining the construction procedure.

THEOREM 5.2.1. *For each k-parallelepiped T_k^s of a given parallelepiped partitioned region D, $0 \leqslant k \leqslant s$, there exists at least one k-vertex spline $f \in V_k^s \subset \hat{S}_d^r$ supported on the union of all the s-parallelepipeds which share T_k^s as the common k-facet.*

Proof. Let us first consider the bivariate case.

(i) *Construction of $V_0^s \subset \hat{S}_d^r$ for $s = 2$.*

Fix a vertex (or 0-parallelepiped) T_0^2 of D. Let T_ν, $\nu = 1, ..., l$, be all those parallelepipeds in D that have T_0^2 as one of their vertices. Write

$T_\nu = \langle T_0^2, \mathbf{w}^\nu, \mathbf{w}^{\nu,1}, \mathbf{w}^{\nu+1}\rangle$, $\nu = 1, ..., l_0$, with the assumption that $\mathbf{w}^{l_0+1} = \mathbf{w}^1$ if T_0^2 is an interior vertex. Suppose that F is a piecewise polynomial function supported on $\bigcup_{\nu=1}^{l_0} T_\nu$ and

$$F|_{T_\nu} = \sum_{\beta \leqslant (d,\, d)} a_\beta^\nu \hat{\phi}_{\nu,\beta}^{(d,d)}, \qquad \nu = 1, ..., l_0.$$

To determine $F \in \hat{S}_d^r$, we specify its Bézier nets a_β^ν on each T_ν via the following steps:

(a) We require that

$$\begin{aligned} D^\beta F(T_0^2) &= c_\beta, && |\beta| \leqslant 2r, \\ D^\beta F(\mathbf{w}^\nu) &= 0, && |\beta| \leqslant 2r,\ \nu = 1, ..., l_0, \textit{ and} \\ D^\beta F(\mathbf{w}^{\nu,1}) &= 0, && |\beta| \leqslant 2r,\ \nu = 1, ..., l_0, \end{aligned} \tag{5.2.1}$$

where $\{c_\beta : |\beta| \leqslant 2r\}$ is a set of real numbers containing at least one nonzero element. Let $N^0 = \{(\beta_1, \beta_2) : \beta_1 + \beta_2 \leqslant 2r\} \cup \{(d-\beta_1, \beta_2) : \beta_1 + \beta_2 \leqslant 2r\} \cup \{(\beta_1, d-\beta_2) : \beta_1 + \beta_2 \leqslant 2r\} \cup \{(d-\beta_1, d-\beta_2) : \beta_1 + \beta_2 \leqslant 2r\}$.

(b) For $F|_{T_\nu}$, we require that

$$\hat{D}^\beta F(T_0^2) = 0, \qquad \beta \in N^1, \qquad \nu = 1, ..., l, \tag{5.2.2}$$

where $\hat{N}^1 = \{(\beta_1, \beta_2) : 2r+1 \leqslant \beta_1 + \beta_2,\ 0 \leqslant \beta_1 \leqslant r,\ 0 \leqslant \beta_2 < d - 2r + \beta_1\}$ and

$$\hat{D}^\beta = D_{\mathbf{w}^\nu - T_0^2}^{\beta_1} D_{\mathbf{w}^{\nu+1} - T_0^2}^{\beta_2}.$$

In addition, we require that

$$D_{\mathbf{w}^\nu - \mathbf{w}^{\nu,1}}^{\beta} D_{\mathbf{w}^{\nu+1} - \mathbf{w}^{\nu,1}}^{\beta_2} F(\mathbf{w}^{\nu,1}) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^1 \tag{5.2.3}$$

and

$$D_{\mathbf{w}^{\nu+1} - \mathbf{w}^{\nu,1}}^{\beta} D_{\mathbf{w}^\nu - \mathbf{w}^{\nu,1}}^{\beta_2} F(\mathbf{w}^{\nu,1}) = 0, \qquad (\beta_1, \beta_2) \in \hat{N}^1. \tag{5.2.4}$$

By applying Theorem 4.2.2, the other interpolation conditions

$$D_{\mathbf{w}^{\nu-1} - T_0^2}^{\beta_1} D_{\mathbf{w}^\nu - T_0^2}^{\beta_2} F(\mathbf{w}^\nu), \qquad (\beta_1, \beta_2) \in \hat{N}^1, \tag{5.2.5}$$

are determined by the corresponding Bézier nets of $F|_{T_{\nu-1}}$, $\nu = 2, ..., l+1$, and we may then use (5.2.5) to determine the corresponding a_β^ν's. Let

$$N^1 = \{(\beta_1, \beta_2), (d-\beta_1, \beta_2), (\beta_1, d-\beta_2), (d-\beta_1, d-\beta_2) : \\ 2r+1 \leqslant \beta_1 + \beta_2,\ 0 \leqslant \beta_1 \leqslant r,\ 0 \leqslant \beta_2 < d - 2r - \beta_1\}.$$

(c) For $F|_{T_\nu}$, $\nu=1, ..., l$, we require that

$$D^{\beta}_{\mathbf{w}^{\nu+1}-\mathbf{w}^{\nu,1}} D^{\beta_2}_{\mathbf{w}^{\nu}-\mathbf{w}^{\nu,1}} F(\mathbf{w}^{\nu,1})=0, \ (\beta_1, \beta_2)\in N^2, \tag{5.2.6}$$

where $N^2=\{(\beta_1, \beta_2): \beta_i \leqslant d, i=1, 2\}\backslash(N^0 \cup N^1)$.

Clearly, by Theorem 3.2.4, we see that $F|_{T_\nu}$ is uniquely determined by the requirements (5.2.1)–(5.2.6). Also, by (5.2.1) it follows that $F\in C^{2r}$ at each vertex in D and by (5.2.1)–(5.2.5) $F\in C^r(D)$. Hence, $F\in \hat{S}^r_d$ and has support $\bigcup_{\nu=1}^{l} T_\nu$; i.e., F is a vertex spline in V_0^2.

(ii) *Construction of* $V_1^2\subset \hat{S}^r_d$ *for* $s=2$.

Fix an edge (or 1-parallelepiped) $T_1^2=\langle \mathbf{w}^1, \mathbf{w}^2\rangle$, and let T_1, T_2 be two parallelograms (or 2-parallelepipeds) sharing T_1^2 as their common 1-facet. Write $T_\nu=\langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^{\nu,3}, \mathbf{w}^{\nu,4}\rangle$, $\nu=1, 2$, and let F be a piecewise polynomial function supported on $T_1\cup T_2$ with

$$F|_{T_\nu}=\sum_{\beta\leqslant(d,d)} a^\nu_\beta \hat{\phi}^{(d,d)}_{\nu,\beta}, \qquad \nu=1,2.$$

To ensure that $F\in\hat{S}^r_d$ we specify the coefficients a^ν_β as follows:

(a) Set

$$D^\beta F(\mathbf{w}^i)=0, \qquad |\beta|\leqslant 2r \tag{5.2.7}$$

for $\mathbf{w}^i\in\{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^{1,3}, \mathbf{w}^{1,4}, \mathbf{w}^{2,3}, \mathbf{w}^{2,4}\}$.

(b) For $F|_{T_1}$, consider the interpolation conditions

$$D^{\beta_1}_{\mathbf{w}^{13}-\mathbf{w}^1} D^{\beta_2}_{\mathbf{w}^2-\mathbf{w}^1} F(\mathbf{w}^1)=c_{(\beta_1,\beta_2)}, \qquad (\beta_1,\beta_2)\in\hat{N}^1, \tag{5.2.8}$$

where $\{c_{(\beta_1,\beta_2)}: (\beta_1,\beta_2)\in\hat{N}^1\}$ is a set of real numbers containing at least one nonzero element. In addition, we require that

$$\begin{aligned}
&D^{\beta_1}_{\mathbf{w}^1-\mathbf{w}^{13}} D^{\beta_2}_{\mathbf{w}^{14}-\mathbf{w}^{13}} F(\mathbf{w}^{13})=0\\
&D^{\beta_1}_{\mathbf{w}^{14}-\mathbf{w}^{13}} D^{\beta_2}_{\mathbf{w}^{1}-\mathbf{w}^{13}} F(\mathbf{w}^{13})=0\\
&D^{\beta_1}_{\mathbf{w}^{13}-\mathbf{w}^{14}} D^{\beta_2}_{\mathbf{w}^{2}-\mathbf{w}^{14}} F(\mathbf{w}^{14})=0\\
&D^{\beta_1}_{\mathbf{w}^{2}-\mathbf{w}^{23}} D^{\beta_2}_{\mathbf{w}^{24}-\mathbf{w}^{23}} F(\mathbf{w}^{23})=0\\
&D^{\beta_1}_{\mathbf{w}^{24}-\mathbf{w}^{23}} D^{\beta_2}_{\mathbf{w}^{2}-\mathbf{w}^{23}} F(\mathbf{w}^{23})=0\\
&D^{\beta_1}_{\mathbf{w}^{23}-\mathbf{w}^{24}} D^{\beta_2}_{\mathbf{w}^{1}-\mathbf{w}^{24}} F(\mathbf{w}^{24})=0
\end{aligned} \tag{5.2.9}$$

for $(\beta_1,\beta_2)\in\hat{N}^1$. See Fig. 5.2.1.

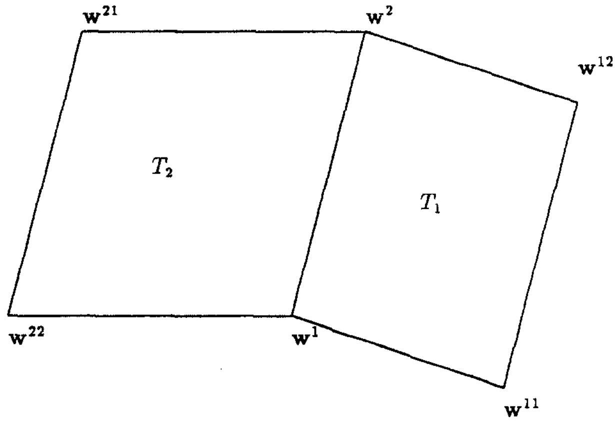

FIGURE 5.2.1

Furthermore, we apply Theorem 4.2.2 to obtain

$$D^{\beta_1}_{w^{24}-w^1} D^{\beta_2}_{w^2-w^1} F(w^1) \tag{5.2.10}$$

from the corresponding coefficients of $F|_{T_1}$ and use (5.2.10) to determine the corresponding a^2_β's.

(c) We require that

$$D^{\beta_1}_{w^1-w^{13}} D^{\beta_2}_{w^{14}-w^{13}} F(w^{13}) = 0, \qquad (\beta_1, \beta_2) \in N^2 \tag{5.2.11}$$

and

$$D^{\beta_1}_{w^2-w^{23}} D^{\beta_2}_{w^{24}-w^{23}} F(w^{23}) = 0, \qquad (\beta_1, \beta_2) \in N^2. \tag{5.2.12}$$

Hence, by Theorem 3.2.4, it is clear that $F|_{T_v}$ is uniquely determined by the requirements (5.2.7)–(5.2.12). It is also clear from (5.2.7) that $F \in C^{2r}$ at each vertex, and that $F \in C^r(D)$ by (5.2.7)–(5.2.10) and Theorem 4.2.2. That is, $F \in \hat{S}^r_d$ and has support given by $T_1 \cup T_2$. In other words, F is a vertex spline in V^2_1.

(iii) *Construction of* $V^2_2 \subset \hat{S}^r_d$ *for* $s = 2$.

Consider a parallelogram (or 2-parallelepiped) T^2_2 in D and suppose that $T^2_2 = \langle w^1, w^2, w^3, w^4 \rangle$ and F is a polynomial supported on T^2_2; that is,

$$F(x) = \begin{cases} \sum_{\beta \leq (d,d)} a_\beta \hat{\phi}^{(d,d)}_\beta, & x \in T^2_2 \\ 0 & \text{otherwise.} \end{cases}$$

To ensure that $F \in \hat{S}_d^r$, we specify its coefficients a_β as follows:

(a) Set

$$D^\beta F(\mathbf{w}^i) = 0, \qquad |\beta| \leqslant 2r,\ i = 1, 2, 3, 4. \tag{5.2.13}$$

(b) For $(\beta_1, \beta_2) \in \hat{N}^1$, specify

$$\begin{aligned} D^{\beta_1}_{\mathbf{w}^4-\mathbf{w}^2} D^{\beta_2}_{\mathbf{w}^1-\mathbf{w}^2} F(\mathbf{w}^2) &= 0 \\ D^{\beta_1}_{\mathbf{w}^4-\mathbf{w}^3} D^{\beta_2}_{\mathbf{w}^1-\mathbf{w}^3} F(\mathbf{w}^3) &= 0 \end{aligned} \tag{5.2.14}$$

and

$$\begin{aligned} D^{\beta_1}_{\mathbf{w}^3-\mathbf{w}^4} D^{\beta_2}_{\mathbf{w}^2-\mathbf{w}^4} F(\mathbf{w}^4) &= 0 \\ D^{\beta_1}_{\mathbf{w}^2-\mathbf{w}^4} D^{\beta_2}_{\mathbf{w}^3-\mathbf{w}^4} F(\mathbf{w}^4) &= 0. \end{aligned}$$

See Fig. 5.2.2 for the orientation of $\{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3, \mathbf{w}^4\}$.

(c) We also require that

$$D^{\beta_1}_{\mathbf{w}^2-\mathbf{w}^4} D^{\beta_2}_{\mathbf{w}^3-\mathbf{w}^4} F(\mathbf{w}^4) = c_{(\beta_1, \beta_2)}, \qquad (\beta_1, \beta_2) \in N^2, \tag{5.2.16}$$

where $\{c_{(\beta_1,\beta_2)}: (\beta_1, \beta_2) \in N^2\}$ is a set of real numbers containing at least one nonzero element.

Clearly, by Theorem 3.2.4, F is uniquely determined by the conditions (5.2.13)–(5.2.16). Also, it follows from (5.2.13) that $F \in C^{2r}$ at each vertex in D and $F \in C^r(D)$ by (5.2.13)–(5.2.15). Hence, $F \in \hat{S}_d^r$; that is, F is a vertex spline in V_2^2.

The procedure in constructing bivariate vertex splines can be generalized to the higher-dimensional setting. We describe the generalization procedure briefly as follows. For a k-parallelepiped T_k^s in D, let $T_1, ..., T_l$ be all those s-parallelepipeds in D that share T_k^s as their common k-facet. Write

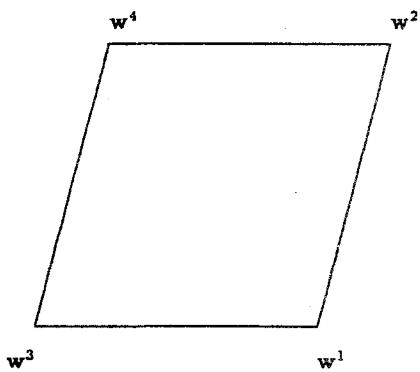

FIGURE 5.2.2

$T_v = \langle \mathbf{w}^{v,1}, ..., \mathbf{w}^{v,e^s} \rangle$, with $\eta^{v,j}$, the index of $\mathbf{w}^{v,j}$ with respect to T_v, $v = 1, ..., l$. Denote by T_{ji}, $i = 1, ..., l_j$, all the j-parallelepipeds of $\{T_v : v = 1, ..., l\}$, $j = 0, ..., s-1$. Let F be a piecewise polynomial function supported on $\bigcup_{v=1}^{l} T_v$, and write

$$F|_{T_v} = \sum_{\beta \leqslant \sigma(d)} a_\beta^v \hat{\phi}_{v,\beta}^{\sigma(d)}, \qquad v = 1, ..., l,$$

where $\sigma(d) = (d, ..., d) \in \mathbf{Z}_+^s$. To ensure that $F \in \hat{S}_d^r$, we specify its Bézier nets a_β^v as follows:

(a) For $j = 0$ and each T_{0i}, $i = 1, ..., l_0$, let T_m, $m \in \{n_{0i,1}, ..., n_{0i,\mu(0i)}\}$ be the s-parallelepipeds in D which have T_{0i} as their common vertex. We require that

$$D^\beta F(T_{0i}) = \begin{cases} 0, & \text{if } \ T_{0i} \neq T_k^s \\ c_\beta, & \text{if } \ T_{0i} = T_k^s \end{cases} \tag{5.2.17}$$

for $|\beta| \leqslant 2^{s-1} r$, where $\{c_\beta : |\beta| \leqslant 2^{s-1} r\}$ is a set of real numbers containing at least one nonzero element. Let $N_{0,j} = \{\beta\eta^j + ((1-n^j)/2)\,\sigma(d),\ |\beta| \leqslant 2^{s-1} r\}$, $j = 1, ..., 2^s$, where η^j denotes the index of $\mathbf{w}^j$ with respect to $T = \langle \mathbf{w}^1, ..., \mathbf{w}^{2s} \rangle$.

(b) For $j = 1, ..., s-1$, and each T_{ji}, $i = 1, ..., l_j$, let T_m, $m \in \{n_{ji,1}, ..., n_{ji,\mu(ji)}\}$, be those s-parallelepipeds T_v, $v = 1, ..., l$ which share T_{ji} as their common j-facet. For T_m, there are $2^{s-j}\binom{s}{s-j}$ j-parallelepiped facets. We enumerate these $2^{s-j}\binom{s}{s-j}$ j-facets by any ordering and denote the uth j-facet by $\langle \mathbf{w}^{u_1}, ..., \mathbf{w}^{u_{2^j}} \rangle$ where $\mathbf{w}^{u_i} = \mathbf{w}^{u_i}(u)$, $i = 1, ..., 2^j$. Then the index η^{u_i} of $\mathbf{w}^{u_i}$, $i = 1, ..., 2^j$, has $s-j$ equal components, say,

$$\eta_{i_v}^{u_1} = \cdots = \eta_{i_v}^{u_{2^j}} = 1 \qquad \text{or} \qquad -1$$

for $v = 1, ..., s-j$, where $1 \leqslant i_v \leqslant s$ since $\langle \mathbf{w}^{u_1}, ..., \mathbf{w}^{u_{2^j}} \rangle$ is a j-facet of T. Hence, we may set

$$N_{j,u} = \left\{ \beta * \eta^{u_1} + \frac{1-\eta^{u_1}}{2} * \sigma(d) : \beta_{i_1} + \cdots + \beta_{i_{s-j}} \leqslant 2^{s-j-1} r \right\}$$

and

$$\hat{N}_{j,u} = N_{j,u} \Big\backslash \bigcup_{p=1}^{j-1} \bigcup_{u=1}^{2^{s-p}\binom{s}{s-p}} N_{p,u}.$$

Fix a T_m, $m \in \{n_{ji,1}, ..., n_{ji,l(ji)}\}$, and assume that $T_{ji} = \langle \mathbf{w}^{m,u_1}, ..., \mathbf{w}^{m,u_{2^j}} \rangle$ for some u, $1 \leqslant u \leqslant 2^{s-j}\binom{s}{s-j}$. Then we require that

$$\hat{D}^\beta F|_{T_m}(\mathbf{w}^{m,u_1}) = \begin{cases} 0 & \text{if } \ T_{ji} \neq T_k^s \\ c_\beta, & \text{if } \ T_{ji} = T_k^s \end{cases} \tag{5.2.18}$$

for $\beta \in \bar{N}_{j,u} = (R_{u_1}^{\sigma(d)})^{-1} \hat{N}_{j,u}$, where $\{c_\beta : \beta \in (R_{u_1}^{\sigma(d)})^{-1} \hat{N}_{j,u}\}$ is a set of real numbers which contains at least one nonzero element. For the other T_p's, $p \in \{n_{ji,1}, ..., n_{ji,l(ji)}\} \backslash m$, we obtain $\hat{D}^\beta F|_{T_p}(\mathbf{w}^{p,u_1(p)})$ from the corresponding coefficients of $F|_{T_m}$ and we use them to determine the coefficients of $F|_{T_p}$ by applying Theorem 4.2.2, where $\beta \in (R_{u_1(p)}^{\sigma(d)})^{-1} \hat{N}_{j,u(p)}$.

(c) For $j = s$ and each $F|_{T_v}$, $v = 1, ..., l$, we require that

$$\hat{D}^\beta F(\mathbf{w}^{v,1}) = \begin{cases} 0, & \text{if} \quad T_v \neq T_k^s \\ c_\beta & \text{if} \quad T_v = T_k^s \end{cases} \tag{5.2.19}$$

for $\beta \in N^s = \{\alpha \in \mathbf{Z}_+^s, \alpha \leqslant \sigma(d)\} \backslash \bigcup_{p=1}^{s-1} \bigcup_{u=1}^{2^{s-p}\binom{s}{s-p}} N_{p,u}$.

By applying Theorem 3.2.4, we see that $F|_{T_v}$ is uniquely determined by the conditions (5.2.17)–(5.2.19) and that $F \in C^{2^{s-j-1}r}$ across each j-dimensional manifold of the partition is confirmed in view of (5.2.17) and Theorem 4.2.2. Therefore, $F \in \hat{S}_d^r$; i.e., F is a vertex spline in V_k^s. Thus, we have completed the proof of the theorem.

It is now easy to construct the basic vertex splines for a given parallelepiped partitioned region D provided that $r \geqslant 0$ and $d \geqslant 2^s r + 1$. The procedure is as follows:

1° For each 0-parallelepiped T_{0i} in D, $i = 1, ..., l_0$, and for each γ with $|\gamma| \leqslant 2^{s-1}r$, let U_{0i}^γ be in $V_0^s \subset \hat{S}_d^r$ with parameters $c_\beta = \delta_{\gamma\beta}$, $|\beta| \leqslant 2^{s-1}r$.

2° For each j, $j = 1, ..., s-1$, and each j-parallelepiped T_{ji} in D, $i = 1, ..., l_j$, and for $\gamma \in (R_{u_1(ji)}^{\sigma(d)})^{-1} \hat{N}_{j,u(ji)}$, let U_{ji}^γ be in $V_j^s \subset \hat{S}_d^r$ with parameter $c_\beta = \delta_{\gamma\beta}$, $\beta \in \bar{N}_{j,u} = (R_{u_1(ji)}^{\sigma(d)})^{-1} \hat{N}_{j,u(ji)}$.

3° For each s-parallelepiped T_v in D, $v = 1, ..., l$, and $\gamma \in N^s$, let U_{sv}^γ be in $V_s^s \subset \hat{S}_d^r$ with parameters $c_\beta = \delta_{\gamma\beta}$, $\beta \in N^s$.

Let $\hat{B}$ be the collection of all vertex splines U_{ji}^γ and U_{sv}^γ constructed as above. Clearly, $\hat{B}$ is a linearly independent set in $\hat{S}_d^r$. Following the same argument as in the proof of Theorem 5.1.2, we have

THEOREM 5.2.2. *For a given parallelepiped partitioned region D, $\hat{B}$ is a basis of $\hat{S}_d^r$.*

To study the approximation order of $\hat{S}_d^r$, let us take a detour by considering the Banach space $C^{k+m}(K)$ with norm $\|v\|_{k+m} = \sum_{|\alpha| \leqslant k+m} \|D^\alpha v\|_\infty$, where $K \subset \mathbf{R}^s$ is a closed and bounded set with Lipschitz continuous boundary. Let $C^{k+m}(K)/\pi_k$ be a quotient space with quotient norm $|\|\cdot\||_{k+m}$ defined, as usual, by

$$|\|\hat{v}\||_{k+m} = \inf_{p \in \pi_k} \{\|\hat{v} + p\|_{k+m}\} \qquad \text{for} \quad \hat{v} \in C^{k+m}/\pi_k.$$

Denote by $|\hat{v}|_{k+j}=\sum_{|\alpha|=k+j}\|D^\alpha v\|_\infty$ for $\hat{v}\in C^{k+m}/\pi_k$. We need the following result, namely which may be used as a substitute for the result of Bramble and Hilbert [6] in proving Theorem 5.1.3. The following lemma is required.

LEMMA. *There exists a constant C such that*

$$|\,\|\hat{v}\|\,|_{k+m}\leqslant C\left(\sum_{j=1}^{m}|\hat{v}|_{k+j}\right)\qquad \textit{for all}\quad \hat{v}\in C^{k+m}/\pi_k.$$

Proof. Let $N=\dim(\pi_k)=\binom{k+s}{s}$ and $\{g_i: 1\leqslant i\leqslant N\}$ be a basis of the dual space of π_k. Let us view g_i, $1\leqslant i\leqslant N$, as linear functionals on $C^{k+m}(D)$ by the Hahn-Banach Extension Theorem. Observe that for a $p\in\pi_k$, we have $g_i(p)=0$, $1\leqslant i\leqslant N$, if and only if $p=0$ since $\{g_i, 1\leqslant i\leqslant N\}$ is a dual basis of π_k. We claim that there exists a constant C such that for all $v\in C^{k+m}(D)$,

$$\|v\|_{k+m}\leqslant C\left(\sum_{j=1}^{m}|v|_{k+j}+\sum_{i=1}^{N}|g_i(v)|\right).$$

Indeed, if this were not true, then there would exist a sequence $\{v_l\}$, v_l in $C^{k+m}(D)$, such that

$$\text{(i)}\quad \|v_l\|_{k+m}=1,\quad \text{and}$$

$$\text{(ii)}\quad \lim_{l\to\infty}\left(\sum_{j=1}^{m}|v_l|_{k+j}+\sum_{i=1}^{N}|g_i(v_l)|\right)=0.$$

Since $\|v_l\|_{k+1}\leqslant\|v_l\|_{k+m}=1$, $\{v_l\}$ is a bounded and equicontinuous family in $C^k(D)$ and by the Ascoli Theorem contains a subsequence $\{v_{l_\nu}\}$ such that

$$\lim_{\nu\to\infty}\|v_{l_\nu}-v_0\|_k=0,$$

where $v_0\in C^k(D)$. Since $\lim_{l\to\infty}|v_l|_{k+1}=0$ by (ii), we see that $\{v_{l_\nu}\}\in C^{k+1}(D)$ is a Cauchy sequence with $\lim_{\nu\to\infty}\|v_{l_\nu}-v_0\|_{k+1}=0$. Therefore, $\|D^\alpha v_0\|_\infty=\lim_{\nu\to\infty}\|D^\alpha v_{l_\nu}\|_\infty=0$, $|\alpha|=k+1$. It follows that $v_0\in\pi_k$. Now, by (ii)

$$g_i(v_0)=\lim_{\nu\to\infty}g_i(v_{l_\nu})=0,\quad 1\leqslant i\leqslant N,$$

which implies that $v_0\equiv 0$. Again, since $\lim_{l\to\infty}\sum_{l=1}^{m}|v_l|_{k+j}=0$, $\{v_{l_\nu}\}$ is a Cauchy sequence in $C^{k+m}(D)$ and $\lim_{\nu\to\infty}\|v_{l_\nu}-v_0\|_{k+m}=\lim_{\nu\to\infty}\|v_{l_\nu}-0\|_{k+m}=0$ and this contradicts (i).

For each $v \in C^{k+m}(D)$, let $p_v \in \pi_k$ such that $g_i(v+p_v)=0$, $1 \leqslant i \leqslant N$. It follows from the above claim that

$$\begin{aligned} \| |\hat{v}| \|_{k+m} &\leqslant \| v+p_v \|_{k+m} \\ &\leqslant C\left(\sum_{j=1}^{m} |v+p_v|_{k+j} + \sum_{j=1}^{N} |g_i(v+p_v)| \right) \\ &= C \sum_{j=1}^{m} |v|_{k+j}. \end{aligned}$$

This completes the proof of the lemma.

With the aid of this lemma, we can verify

THEOREM 5.2.3. *Suppose that* $f \in C^{sd}(D)$, $d \geqslant 2^s r + 1$, *and* $s > 1$. *Then*

$$\inf_{s \in \hat{S}^r_d} \| f - s \|_\infty \leqslant C h^{d+1} \max_{d+1 \leqslant |\alpha| \leqslant sd} \| D^\alpha f \|_\infty,$$

where h is the maximum of the diameters of all parallelepipeds in D and C is a constant independent of f and h.

Proof. Let us define a map $M: C^{sd}(D) \to \hat{S}^r_d(D)$ by

$$\begin{aligned} Mf(\mathbf{x}) = &\sum_{i=1}^{l_0} \sum_{|\gamma| \leqslant 2^{s-1} r} D^\gamma f(T_{0i})\, U^\gamma_{0i}(\mathbf{x}) \\ &+ \sum_{j=1}^{s-1} \sum_{i=1}^{l_j} \sum_{\gamma \in \hat{N}_{j,u(ji)}} \hat{D}^\gamma f(\mathbf{w}^{m_{ji}, u_1(ji)})\, U^\gamma_{ji}(\mathbf{x}) \\ &+ \sum_{\nu=1}^{l} \sum_{\gamma \in N^s} \hat{D}^\gamma f(\mathbf{w}^{\nu,1})\, U^s_{s\nu}(\mathbf{x}). \end{aligned}$$

Clearly, $M: C^{sd}(D) \to \hat{S}^r_d(D)$ is an interpolation operator and it can be shown that $Mp = p$ for any $p \in \pi_d$ by verifying that Mp interpolates p on each parallelepiped, using induction and applying Theorem 4.2.3.

For a fixed $\mathbf{x} \in D$, consider

$$F(f) = f(\mathbf{x}) - Mf(\mathbf{x}).$$

It is clear that $F(f)$ satisfies

(i) $F(f)| \leqslant C_1 \sum_{l=0}^{sd} h^l |f|_l$ for some constant C_1 independent of f and h and

(ii) $F(p) = 0$ for all $p \in \pi_d$.

Let us first assume that $h=1$. Clearly,

$$|F(f)|=|F(f+p)|\leqslant C_1 \sum_{l=0}^{sd} |f+p|_l = C_1 \| f+p \|_{sd}$$

for any $p\in\pi_d$. It follows that

$$|F(f)|\leqslant C_1 \, ||| \hat{f} |||_{sd}.$$

By the above lemma,

$$|F(f)|\leqslant C_1 \, ||| \hat{f} |||_{sd}\leqslant C_2 \sum_{j=1}^{(s-1)d} |f|_{d+j} = C \max_{d+1\leqslant|\beta|\leqslant sd} \| D^\beta f \|_\infty .$$

Now for any $h>0$, we simply let $x=hy$, $g(y)=f(hy)$, and $\tilde{D}=\{y: hy\in D\}$. Then the maximum of the diameters of all parallelepipeds of $\tilde{D}$ induced from that of D is 1. Thus,

$$\begin{aligned} |F(f)| = |\tilde{F}(g)| &\leqslant C_2 \sum_{j=1}^{(s-1)d} |g|_{d+j} \\ &= C_2 \sum_{j=1}^{(s-1)d} h^{d+j} |f|_{d+j} \\ &\leqslant Ch^{d+1} \max_{d+1\leqslant|\beta|\leqslant sd} \| D^\beta f \|_\infty \end{aligned}$$

which completes the proof of the theorem.

5.3. *Mixed partitioned regions in* $\mathbf{R}^2$

Let D be a mixed partitioned region in $\mathbf{R}^2$. We first prove the existence result by outlining the construction procedure.

THEOREM 5.3.1. *Let $d\geqslant 4r+1$, $r\geqslant 0$. For each vertex (or edge) of D, there exists at least one vertex spline in $\hat{S}^r_d$ with support given by the union of those cells (triangles or parallelograms) which share the given vertex (or edge). In addition, for $r\geqslant 2$ and any given cell (triangle or parallelogram), there exists at least one vertex spline in $\hat{S}^r_d$ whose support is this given cell. However, there is no nontrivial function V^2_2 in $\hat{S}^1_5(D)$ whose support is a single triangle.*

Proof. (i) *Construction of $V^2_0\subset\hat{S}^r_d$.*

Let V be a vertex in the mixed partitioned region D and let S_v, $v=1, ..., l$, be the cells (triangles or parallelograms) in D which have V as one of their vertices. Let T^0_i, $i=1, ..., l_0$, and T^1_i, $i=1, ..., l_1$, be all the vertices and edges of $\bigcup_{v=1}^{l} D_v$, respectively, and F be a piecewise polynomial supported on $\bigcup_{v=1}^{l} S_v$ such that

$$F|_{S_v} = \begin{cases} \sum_{|\alpha|=d} a_\alpha^v \phi_\alpha^d & \text{if } S_v \text{ is a triangle} \\ \sum_{\beta \leqslant (d,d)} b_\beta^v \hat{\phi}_\beta^{(d,d)} & \text{if } S_v \text{ is a parallelogram.} \end{cases}$$

To ensure that $F \in \hat{S}_d^r$, we specify the Bézier nets of $F|_{S_v}$, $v = 1, ..., l$, as follows:

(a) For T_i^0, $i = 1, ..., l_0$, we require that

$$D^\beta F(T_i^0) = \begin{cases} 0 & \text{if } T_i^0 \neq V \\ c_\beta & \text{if } T_i^0 = V, \end{cases} \tag{5.3.1}$$

for $|\beta| \leqslant 2r$, where $\{c_\beta : |\beta| \leqslant 2e\}$ is a parameter set of real numbers which are not all equal to zero.

(b) For each T_i^1, $i = 1, ..., l_1$, there are four cases to be considered: (1°) only one cell intersects with T_i^1; (2°) two triangles share T_i^1; (3°) two parallelograms share T_i^1; and (4°) one triangle S_{v_1} and one parallelogram S_{v_2} share T_i^1. For the first three cases, our interpolation conditions of F are the same as those in the proofs of Theorems 5.1.1 and 5.2.1. For the final case, we let $T_i^1 = \langle \mathbf{w}^1, \mathbf{w}^2 \rangle$ and require that $F|_{S_{v_1}}$ satisfy

$$D_{\mathbf{w}^2 - \mathbf{w}^1}^{\beta_1} D_{\mathbf{v} - \mathbf{w}^1}^{\beta_2} F(\mathbf{w}^1) = 0 \tag{5.3.2}$$

for $(\beta_1, \beta_2) \in \{(\beta_1, \beta_2);\ 0 \leqslant \beta_2 \leqslant r, 0 \leqslant \beta_1 < d - 2r, \beta_1 + \beta_2 > 2r\}$, where $S_{v_1} = \langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{v} \rangle$. Also, we obtain, by using Theorem 4.3.1,

$$D_{\mathbf{w}^2 - \mathbf{w}^1}^{\beta_1} D_{\mathbf{v}^1 - \mathbf{w}^1}^{\beta_2} F|_{S_{v_2}}(\mathbf{w}^1), \tag{5.3.3}$$

where $(\beta_1, \beta_2) \in \{(\beta_1, \beta_2) : 0 \leqslant \beta_2 \leqslant r,\ 2r \leqslant \beta_1 + \beta_2,\ 0 \leqslant \beta_1 < d - 2r + \beta_2\}$ from the corresponding coefficients of $F|_{S_{v_1}}$ and use (5.3.3) to determine appropriate coefficients of $F|_{S_{v_2}}$.

(c) For each S_v, $v = 1, ..., l$, there are two cases to be considered: (1°) S_v is a triangle and (2°) D_v is a single parallelogram. Our interpolation conditions on S_v for cases (1°) and (2°), are the same as those in the proofs of Theorems 5.1.1 and 5.2.1, respectively. Clearly, $F|_{S_v}$ is uniquely determined by the conditions (a) through (c) by the application of Theorems 3.1.5 and 3.2.4. That $F \in C^{2r}$ at each vertex follows by observing (5.3.1) and that $F \in C^r(D)$ may be confirmed by applying Theorems 4.1.2, 4.2.2, and 4.3.1. Hence, F is a vertex spline in $\hat{S}_d^r$.

(ii) *Construction of* $V_1^2 \subset \hat{S}_d^r$

Let $T = \langle \mathbf{w}^1, \mathbf{w}^2 \rangle$ be an edge of D and let S_{v_1}, S_{v_2} be two cells that share T. Then there are three cases to be considered: (1°) S_{v_1}, S_{v_2} are two triangles; (2°) S_{v_1}, S_{v_2} are two parallelograms; and (3°) S_{v_1} is a triangle and S_{v_2} a parallelogram. For the first two cases, we have shown the construction of V_1^2 with support given by $S_{v_1} \cup S_{v_2}$ as in the proofs of Theorems 5.1.1 and 5.2.1. For case (3°), let F be a piecewise polynomial function supported on $S_{v_1} \cup S_{v_2}$ with

$$F|_{S_{v_1}} = \sum_{|\alpha| = d} a_\alpha \phi_\alpha^d,$$

$$F|_{S_{v_2}} = \sum_{\beta \leqslant (d,d)} b_\beta \phi_\beta^{(d,d)},$$

where $S_{v_1} = \langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{v}^3 \rangle$ and $S_{v_2} = \langle \mathbf{w}^1, \mathbf{w}^2, \mathbf{v}^1, \mathbf{v}^2 \rangle$. To ensure that $F \in \hat{S}_d^r$, we specify the Bézier nets of $F|_{S_{v_2}}$ and $F|_{S_{v_2}}$ as follows:

(a) For $\mathbf{v} \in \{\mathbf{w}^1, \mathbf{w}^2, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$, we require that

$$D^\beta F(\mathbf{v}) = 0, \qquad |\alpha| \leqslant 2r.$$

(b) For $F|_{S_{v_1}}$, we require that

$$D^{\beta_1}_{\mathbf{w}^2-\mathbf{w}^1} D^{\beta_2}_{\mathbf{v}^3-\mathbf{w}^1} F(\mathbf{w}^1) = c_\beta$$

for $(\beta_1, \beta_2) \in N^1 = \{(\beta_1, \beta_2): 0 \leqslant \beta_2 \leqslant r,\ \beta_1 + \beta_2 > 2r,\ 0 \leqslant \beta_1 \leqslant d - 2r - 1\}$, where the c_β's are parameters which are not all equal to zero. We may then determine $b_{(j,k)}$, $0 \leqslant k \leqslant r$, $2r - k < j < d - 2r + k$ of $F|_{S_{v_2}}$. Also, we impose the conditions

$$D^{\beta_1}_{\mathbf{w}^1-\mathbf{v}^3} D^{\beta_2}_{\mathbf{w}^2-\mathbf{v}^3} F(\mathbf{v}^3) = 0$$

and

$$D^{\beta_1}_{\mathbf{w}^2-\mathbf{v}^3} D^{\beta_2}_{\mathbf{w}^1-\mathbf{v}^3} F(\mathbf{v}^3) = 0$$

for $(\beta_1, \beta_2) \in N^1$. For $F|_{S_{v_2}}$, we require that

$$D^{\beta_1}_{\mathbf{w}^1-\mathbf{v}^1} D^{\beta_2}_{\mathbf{v}^2-\mathbf{v}^1} F(\mathbf{v}^1) = 0,$$
$$D^{\beta_1}_{\mathbf{v}^2-\mathbf{v}^1} D^{\beta_2}_{\mathbf{w}^1-\mathbf{v}^1} F(\mathbf{v}^1) = 0,$$

and

$$D^{\beta_1}_{\mathbf{w}^2-\mathbf{v}^2} D^{\beta_2}_{\mathbf{v}^1-\mathbf{v}^2} F(\mathbf{v}^2) = 0,$$

for $(\beta_1, \beta_2) \in N^1$.

(c) For $F|_{S_{v_1}}$, we require that

$$D^{\beta_1}_{\mathbf{w}^1-\mathbf{v}^3} D^{\beta_2}_{\mathbf{w}^2-\mathbf{v}^3} F(\mathbf{v}^3) = 0$$

for $(\beta_1, \beta_2) \in N^2 = \{(\beta_1, \beta_2); 2r < \beta_1 + \beta_2 < d-r,\ r < \beta_1 < d-2r,\ r < \beta_2 < d-2r\}$. For $F|_{S_{v_1}}$, we require that

$$D^{\beta_1}_{\mathbf{v}^2-\mathbf{v}^1} D^{\beta_2}_{\mathbf{w}^1-\mathbf{v}^1} F(\mathbf{v}^1) = 0$$

for $(\beta_1, \beta_2) \in N^3 = \{(\beta_1, \beta_2), r < \beta_1 < d-r, r < \beta_2 < d-r\}$.

Clearly, by Theorems 3.1.5 and 3.2.4, $F|_{S_{v_2}}$ is uniquely determined by the conditions above. Also, F is in C^{2r} at all the vertices by the requirement in (a), and that $F \in C^r(D)$ may be confirmed by the condition (b) and by applying Theorem 4.3.1. Therefore, $F \in \hat{S}^r_d$.

(iii) *Construction of* $V^2_2 \subset \hat{S}^r_d$

Let T be a cell of D. Then T is either a triangle or a parallelogram. The construction of a vertex spline on T is similar to that given in the proofs of Theorem 5.1.1 or Theorem 5.2.1, respectively. This completes the proof of the theorem.

Let us now construct the basic vertex splines for a given mixed partitioned region $D = \bigcup_{v=1}^{l} S_v$ in $\mathbf{R}^2$ as follows:

1° For each vertex T_{0i}, $i = 1, ..., l_0$, of the partitioned region D and $\gamma \in \mathbf{Z}^2_+$ with $|\gamma| \leqslant 2r$, let V^2_{0i} be a function in $V^2_0 \subset \hat{S}^r_d(D)$ supported on the union of cells of D that have T_{0i} as one of their vertices with parameters $c_\beta = \delta_{\beta\gamma}$, $|\beta| \leqslant 2r$.

2° For each edge T_{1i}, $i = 1, ..., l_1$ of (the partition of) D and $\gamma = (\gamma_1, \gamma_2) \in N^1$, let V^γ_{1i} be a $V^2_1 \subset \hat{S}^r_d(D)$ supported on the union of cells of D that share T_{1i} with parameters $c_\beta = \delta_{\beta\gamma}$, $\beta \in N^1$.

3° For each triangle T_{2i} and $\gamma \in N^2$, let V^γ_{2i} be in $V^2_2 \subset \hat{S}^r_d(D)$ and supported on T_{2i} with parameters $c_\beta = \delta_{\beta\gamma}$, $\beta \in N^2$; and for each parallelogram T'_{2i}, and $\gamma \in N^3$, let V^γ_{2i} be a function in $V^2_2 \subset \hat{S}^r_d(D)$ supported on T'_{2i} with parameters $c_\beta = \delta_{\beta\gamma}$, $\beta \in N^3$.

Let $\tilde{B}$ be the collection of all basic vertex splines so constructed. Then the following results can be derived in the same manner as before.

THEOREM 5.3.2. *For any given mixed partition region D, $\tilde{B}$ provides a basis of $\hat{S}^r_d(D)$.*

THEOREM 5.3.3. *Suppose that $f \in C^{2d}(D)$, $d \geqslant 4r+1$. Then*

$$\inf_{s \in S^r_d} \|f - s\|_\infty \leqslant Ch^{d+1} \max_{d+1 \leqslant |\beta| \leqslant 2d} \|D^\beta f\|_\infty,$$

where h is the maximum of the diameters of the triangles or parallelograms of D and C is a constant independent of f and h.

6. Applications to L^2 and l^2 Approximation with Interpolatory Constraints

We now apply the vertex splines developed in Sections 5.1, 5.2, and 5.3 to least-squares approximation with interpolatory constraints. Assume that $D \subset \mathbf{R}^s$ is a simplicial partitioned region or parallelepiped partitioned region (or mixed partitioned region if $D \subset \mathbf{R}^2$). Let V denote the set of all vertices of D and $I = \{\alpha \in \mathbf{Z}_+^s : |\alpha| \leqslant 2^{s-1}r\}$ and I_c a subset of $V \times I$ which we will call an index set for interpolatory constraints. Note that I_c may be empty. The problems of L^2 or l^2 approximation with interpolatory constraints can be stated as follows:

(L^2-I_c) Given a function $f: D \to \mathbf{R}$, find the super spline $S_f \in \hat{S}_d^r(D)$, where $d \geqslant 2^s r + 1$, $r \geqslant 0$, such that

$$\| f - S_f \|_{D,2} = \inf\{ \| f - s \|_{D,2} : s \in \hat{S}_d^r \text{ and } D^\alpha s(\mathbf{v}) = D^\alpha f(\mathbf{v}), (\mathbf{v}, \alpha) \in I_c\}. \quad (6.1)$$

Here $\| g \|_{D,2} = (\int_D | g(\mathbf{x})|^2 \, d\mathbf{x})^{1/2}$. Note that when $I_c = \varnothing$, Problem (6.1) is the usual L^2 approximation problem. (See [8] for example.)

(l^2-I_c) Given only sample data $\{(\mathbf{y}_i, f(\mathbf{y}_i), w_i), i = 1, ..., L\}$ with weights $w_i > 0$, $i = 1, ..., L$, where $Y = \{\mathbf{y}_i\}_i^L \subset D$ such that if any $(\mathbf{v}, \alpha) \in I_c$, then $\mathbf{v} \in Y$, find a super spline $s_f \in \hat{S}_d^r(D)$, $d \geqslant 2^s r + 1$, $\geqslant 0$, such that

$$\| f - s_f \|_{2,\mathbf{w}} = \inf\{ \| f - s \|_{2,\mathbf{w}} : s \in \hat{S}_d^r \text{ and } D^\alpha s(\mathbf{v}) = D^\alpha f(\mathbf{v}), (\mathbf{v}, \alpha) \in I_c\} \quad (6.2)$$

and give a uniqueness criterion. Here, $\| f \|_{2,\mathbf{w}} = (\sum_{i=1}^L w_i \, | f(\mathbf{y}_i)|^2)^{1/2}$. The weights $\mathbf{w} = \{w_i\}$ may be normalized so that $\sum_{i=1}^L w_i = 1$. Usually, the quantity of data $\{\mathbf{y}_i . f(\mathbf{y}_i), w_i\}_1^L$ is very large so that we will always assume that $L \geqslant M$, where M denotes the dimension of $\hat{S}_d^r(D)$. Note that when $I_c = \varnothing$, the problem becomes usual l^2 approximation.

Denote by V_i, $i = 1, ..., M$, all the basic vertex splines in $\hat{S}_d^r(D)$ constructed in Section 5.1, 5.2, or 5.3 accordingly. Also, let $V_{0,\mathbf{v}}^\alpha$ be the basic vertex splines in V_0^s that satisfy $D^\gamma V_{0,\mathbf{v}}^\alpha(\mathbf{u}) = \delta_{\mathbf{v},\mathbf{u}} \delta_{\alpha,\gamma}$, where $(\mathbf{v}, \alpha) \in I_c$. For simplicity, we rearrange if necessary so that $\{V_i : i = 1, ..., M - m\} = \{V_i : i = 1, ..., M\} \backslash \{V_{0,\mathbf{v}}^\alpha : (\mathbf{v}, \alpha) \in I_c\}$, where $m = \# I_c$ is the cardinalty of the index set I_c. Then, clearly Problem (L^2-I_c) is equivalent to solving the linear system

$$[A_{ij}] \mathbf{c} = \mathbf{b}, \quad (*)$$

where $A_{ij} = \int_D V_i(\mathbf{x}) \, V_j(\mathbf{x}) \, d\mathbf{x}$, $i, j = 1, ..., M - m$, $\mathbf{c} = (c_1, ..., c_{M-m})^T$, and $\mathbf{b} = (b_1, ..., b_{M-m})^T$ with

$$b_i = \int_D \left(f(\mathbf{x}) - \sum_{(\mathbf{v}, \alpha) \in I_c} D^\alpha f(\mathbf{v}) \, V_{0,\mathbf{v}}^\alpha(\mathbf{x}) \right) V_i(\mathbf{x}) \, d\mathbf{x}.$$

Observing that $V_i(\mathbf{x})$, $i = 1, ..., M-m$, are linearly independent, we note that the $(M-m) \times (M-m)$ Gramian matrix $[A_{ij}]$ is nonsingular and $(*)$ has a unique solution $\mathbf{c} = (c_1, ..., c_{M-m})^T$. We also note that A_{ij} can be easily computed by using Lemma 2.1.2 or Lemma 2.2.2, and b_i may be estimated by using some quadrature formula in numerical computation. We state this simple result for completeness.

THEOREM 6.1. *Problem (L^2-I_c) has a unique solution S_f in the super spline space $\hat{S}^r_d(D)$, $d \geq 2^s r + 1$ and $r \geq 0$, where*

$$S_f(\mathbf{x}) = \sum_{(\mathbf{v}, \alpha) \in I_c} D^\alpha f(\mathbf{v}) V^\alpha_{0,\mathbf{v}}(\mathbf{x}) + \sum_{i=1}^{M-m} c_i V_i(\mathbf{x})$$

with $\mathbf{c} = (c_1, ..., c_M)^T = [A_{ij}]^{-1} \mathbf{b}$.

By using Theorem 5.1.3, Theorem 5.2.3, or Theorem 5.3.3, we easily obtain

THEOREM 6.2. *Let $d \geq 2^s r + 1$ and consider $f \in C^{d+1}(D)$ if D is a simplicial partitioned region or $f \in C^{sd}(D)$ if D is a parallelepiped partitioned region in $\mathbf{R}^s$ (or a mixed partitioned region in $\mathbf{R}^2$). Then*

$$\| f - S_f \|_{D,2} \leq C h^{d+1},$$

where C depends only on the function f.

We now turn to the study of Problem (l^2-I_c). Again, let $V_i(\mathbf{x})$, $i = 1, ..., M-m$ be the basic vertex splines in $\hat{S}^r_d(D)$ as above. We will use the notation

$$\boldsymbol{l}_i = (V_i(\mathbf{y}_1), ..., V_i(\mathbf{y}_L))^T, \qquad i = 1, ..., M-m,$$

and $\mathbf{f} = (f(\mathbf{y}_1), ..., f(\mathbf{y}_L))^T$. Further, let

$$\tilde{f} = f - \sum_{(\mathbf{v}, \alpha) \in I_c} D^\alpha f(\mathbf{v}) V^\alpha_{0,\mathbf{v}}(\mathbf{x})$$

and $\tilde{\mathbf{f}} = (\tilde{f}(\mathbf{y}_1), ..., \tilde{f}(\mathbf{y}_L))^T$. Clearly, Problem ($l^2$-$I_c$) can be reformulated as follows. Determine $\mathbf{c} = (c_1, ..., c_{M-m})^T$ such that

$$\left\| \tilde{\mathbf{f}} - \sum_{i=1}^{M-m} c_i \boldsymbol{l}_i \right\|_{2,\mathbf{w}} = \inf_{(a_1, ..., a_{M-m})^T} \left\| \tilde{\mathbf{f}} - \sum_{i=1}^{M-m} a_i \boldsymbol{l}_i \right\|_{2,\mathbf{w}}. \tag{6.3}$$

Since $\boldsymbol{l}_i$, $i = 1, ..., M-m$, are not necessarily independent, Problem (l^2-I_c) may have more than one solution. Following Hayes [18], we give a uniqueness criterion as follows: Let X be the set of solutions to (6.3). Then we consider the following "adaptive" l^2-approximation problem:

$(l^2\text{-}I_c)'$ Determine $\hat{s}_f=\sum_{i=1}^{M-m} c_i V_i+\sum_{(\mathbf{v},\alpha)\in I_c} D^\alpha f(\mathbf{v})\, V^\alpha_{0,\mathbf{v}}\in \hat{S}^r_d$ that satisfies (6.2) and

$$\left(\sum_{i=1}^{M-m} |c_i|^2\right)^{1/2}=\inf\left\{\left(\sum_{i=1}^{M-m} |a_i|^2\right)^{1/2}, (a_1, ..., a_{M-m})^T\in X\right\}. \quad (6.4)$$

Then we have the following result.

THEOREM 6.3. *Problem* $(l^2\text{-}I_c)'$ *has a unique solution in* $\hat{S}^r_d$, *where* $d\geq 2^s r+1$.

Proof. Let

$$\begin{aligned}\bar{Y}&=\left\{\left(\sum_{i=1}^{M-m} a_i V_i(\mathbf{y}_1), ..., \sum_{i=1}^{M-m} a_i V_i(\mathbf{y}_L)\right): (a_i, ..., a_{M-m})^T\in X\right\}\\ &=\{(\boldsymbol{l}_1\cdots \boldsymbol{l}_{M-m})(a_1, ..., a_{M-m})^T: (a_1, ..., a_{M-m})^T\in X\}\end{aligned}$$

and η^j, $j=1, ..., k$, be a basis of the null space of $(\boldsymbol{l}_1\cdots \boldsymbol{l}_{M-m})$. Then it follows that

$$\bar{Y}=\{(\boldsymbol{l}_1\cdots \boldsymbol{l}_M)(\eta^*+\alpha_1\eta^1+\cdots+\alpha_k\eta^k): \alpha_1, ..., \alpha_k\in \mathbf{R}\}$$

where $\eta^*=(a_1^*, ..., a_{M-m}^*)^T\in X$.

Hence, it follows that (6.4) is equivalent to

$$\left(\sum_{i=1}^{M-m} |c_i|^2\right)^{1/2}=\min_{\alpha_1, ..., \alpha_k} \|\eta^*+\alpha_1\eta^1+\cdots+\alpha_k\eta^k\|_{l^2}$$

which will give a unique solution, since $\eta^1, ..., \eta^k$ are linearly independent. This completes the proof of the theorem.

Actually, as is well known, Problem $(l^2\text{-}I_c)'$ may be solved by using the Moore–Penrose pseudoinverse; that is, $\hat{s}_\mathbf{f}=\sum_{(\mathbf{v},\alpha)\in I_c} D^\alpha f(\mathbf{v})\, V^\alpha_{0,\mathbf{v}}+\sum_{i=1}^{M-m} c_i V_i$, where $\mathbf{c}=(c_1, ..., c_{M-m})^T$ is the limit of

$$((\boldsymbol{l}_1\cdots \boldsymbol{l}_{M-m})^*(\boldsymbol{l}_1\cdots \boldsymbol{l}_{M-m})+\varepsilon I)^{-1}(\boldsymbol{l}_1\cdots \boldsymbol{l}_{M-m})^*\,\bar{\mathbf{f}}$$

as $\varepsilon\to 0^+$ and I is the identity matrix (cf. Luenberger [23]).

The important question is how well $\hat{s}_f$ approximates $\mathbf{f}$. The answer is somewhat delicate since $\hat{s}_f$ does not necessarily converge to f as the number of sample data increases and the size of the simplices or parallelepipeds decreases to zero. However, if the sample data are fairly *dense* on a subset E of D, we may still expect $\hat{s}_f$ to be close to f on E. In this respect, we need the notation

$$d_E=\max_{x\in E}\min_{1\leq i\leq L} |\mathbf{x}-\mathbf{y}^i|.$$

In addition, set

$$\delta_E = \min\{w_i : \mathbf{y}^i \in E\}$$

and let Δ_E be the minimum of the radii of the balls inscribed in the (simplicial or parallelepiped) cells that have nonempty intersection with E.

Let $E \subset D$ be a subdomain which is the union of some parallelepipeds d_i, $i = 1, ..., \hat{L}$, that are parallel to the coordinate hyperplanes and each of which contains at least one $\mathbf{y}^i \in E$. We also need the constant $C(d)$ of the Markoff inequality on multivariate polynomials in the L^2 norm. This is defined by

$$C(d) = \max_{\substack{\|p_d\|_\Omega = 1 \\ i = 1, ..., s}} \left\| \frac{\partial}{\partial x_i} p_d \right\|_\Omega,$$

where Ω is either the standard simplex $\langle 0, e^1, ..., e^s \rangle$ with $e^i = (0, ..., 0, 1, 0, ..., 0)$ or the unit cube $[0, 1]^s$, and the maximum is taken over all norm-one polynomials p_d of degree d, which may be the total degree or coordinate degree depending on Ω.

We are now ready to state the next result.

THEOREM 6.4. *Let E be a subset of D with*

$$C(d)\, d_E^{s/2} < \Delta_E.$$

Then for any $f \in C^{d+1}(D)$ if D is a simplicial partitioned region, or $f \in C^{sd}(D)$ if D is a parallelepiped partitioned region (or a mixed partitioned region in $\mathbf{R}^2$),

$$\| f - \hat{s}_\mathbf{f} \|_{E,2} \leqslant K \left(1 - \frac{C(d)}{\Delta_E} d_E^{s/2} \right)^{-1} (\delta_E)^{-1/2} h^{d+1},$$

where $\mathbf{f} = \{f(\mathbf{y}_i)\}$, $i = 1, ..., L$, $\hat{s}_\mathbf{f}$ is the unique solution of Problem $(l^2 - I_c)'$, and the constant K depends only on f.

Proof. Let

$$s_\mathbf{f} = \hat{s}_\mathbf{f} - \sum_{(\mathbf{v}, \alpha) \in I_c} D^\alpha f(\mathbf{v})\, V^\alpha_{0, \mathbf{v}}.$$

Since

$$\begin{aligned}
\| s - s_\mathbf{f} \|_{E,2} &\leqslant \left(\sum_{i=1}^{\hat{L}} \int_{d_i} |(s - s_\mathbf{f})(\mathbf{x})|^2\, d\mathbf{x} \right)^{1/2} \\
&= \left(\sum_{i=1}^{\hat{L}} |(s - s_\mathbf{f})(\xi_i)|^2 \operatorname{vol}(d_i) \right)^{1/2} \\
&\leqslant \| s - s_\mathbf{f} \|_{Y \cap E, \mathbf{w}}\, \delta_E^{-1/2} \\
&\quad + \left(\sum_{i=1}^{\hat{L}} |(s - s_\mathbf{f})(\xi_i) - (s - s_\mathbf{f})(\mathbf{y}_{n_i})|^2 \operatorname{vol}(d_i) \right)^{1/2}
\end{aligned}$$

$$= \| s - s_{\mathbf{f}} \|_{Y \cap E, \mathbf{w}} \delta_E^{-1/2} + \left(\sum_{i=1}^{\hat{L}} \left| \int_{\mathbf{y}_{n_i}}^{\xi_i} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_s} (s - s_{\mathbf{f}})(\mathbf{t}) \, dt \right|^2 \operatorname{vol}(d_i) \right)^{1/2}$$

$$\leqslant \| s - s_{\mathbf{f}} \|_{Y \cap E, \mathbf{w}} \delta_E^{-1/2} + d_E^{s/2} \left(\sum_{T_i \cap E \neq \varnothing} \int_{T_i} \left| \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_s} (s - s_{\mathbf{f}}) \right|^2 dt \right)^{1/2},$$

we have

$$\| s - s_{\mathbf{f}} \|_{E,2} \leqslant \| s - s_{\mathbf{f}} \|_{Y \cap E, \mathbf{w}} \delta_E^{-1/2} + \frac{C(d)\, d_E^{s/2}}{\Delta_E} \| s - s_{\mathbf{f}} \|_{E,2},$$

or

$$\| s - \mathbf{s}_{\mathbf{f}} \|_{E,2} \leqslant \left(1 - \frac{C(d)}{\Delta_E} d_E^{s/2} \right)^{-1} \delta_E^{-1/2} \| s - s_{\mathbf{f}} \|_{Y \cap E, \mathbf{w}}.$$

Hence,

$$\| f - \mathbf{s}_{\mathbf{f}} \|_{E,2} \leqslant \| f - s \|_{E,2} + \| s - s_{\mathbf{f}} \|_{E,2}$$

$$\leqslant \operatorname{vol}(E) \| f - s \|_{E,\infty} + \left(1 - \frac{C(d)}{\Delta_E} d_E^{s/2} \right)^{-1} \delta_E^{-1/2} \left(\| s - f \|_{Y \cap E, 2} + \| f - s_{\mathbf{f}} \|_{Y \cap E, 2} \right)$$

$$\leqslant \operatorname{vol}(E) \| f - s \|_{D,\infty} + 2 \left(1 - \frac{C(d)}{\Delta_E} d_E^{s/2} \right)^{-1} \delta_E^{-1/2} \| s - f \|_{Y,2}$$

which, in view of Theorem 5.1.2, Theorem 5.2.3, or Theorem 5.3.3, yields the desired result.

Remark. We may generalize the above study to L^p and l^p approximation, $1 \leqslant p \leqslant \infty$, and similar results can be established.

7. Examples of Vertex Splines

For simplicity, we consider only examples of vertex splines in $\hat{S}_5^1$ in $\mathbf{R}^2$ and present their polynomial pieces in terms of the Bézier nets (see Figs. 7.1–7.7). Pictures of these vertex splines on various supports are also included in this section (see Figs. 7.8–7.35).

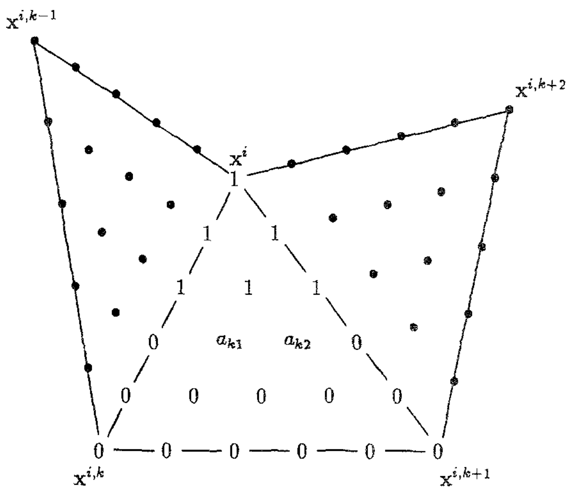

FIG. 7.1. 0-Vertex spline $V_{0i}^{(0,0)}$

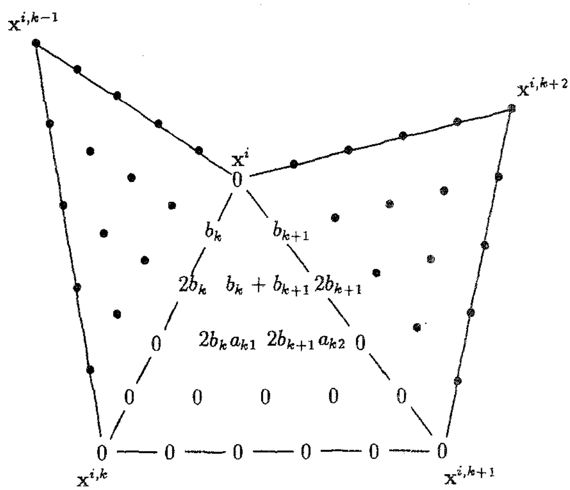

FIG. 7.2. 0-Vertex spline $V_{0i}^{(1,0)}$

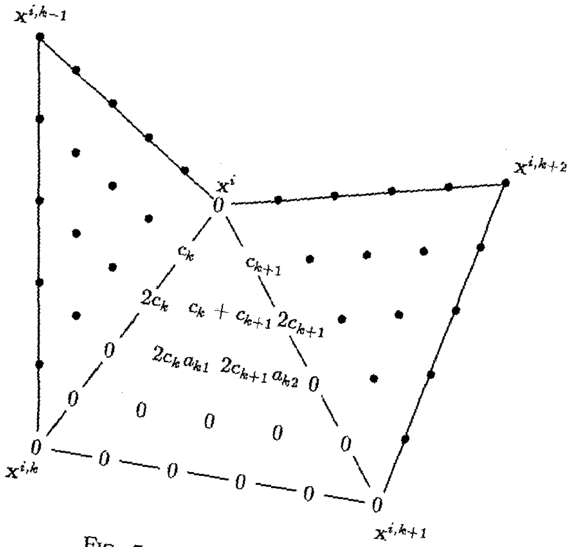

FIG. 7.3. 0-Vertex spline $V_{0i}^{(0,1)}$

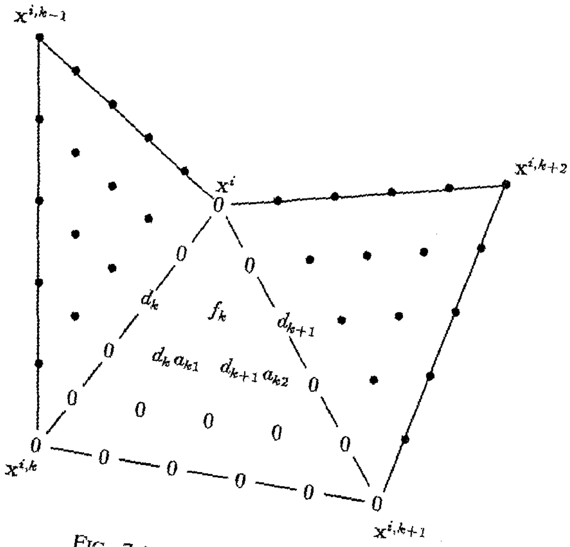

FIG. 7.4. 0-Vertex spline $V_{0i}^{(2,0)}$

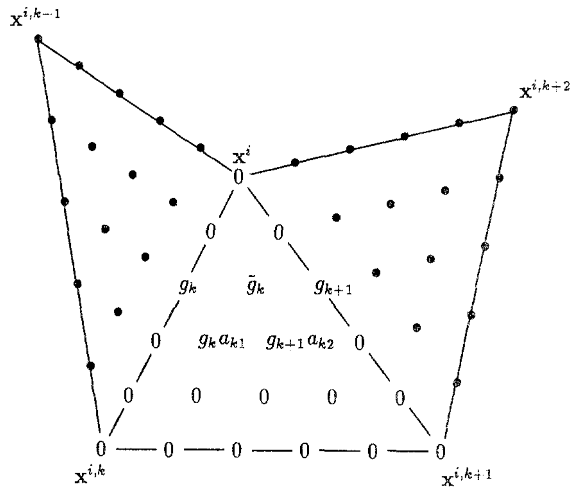

FIG. 7.5. 0-Vertex spline $V_{0i}^{(1,1)}$

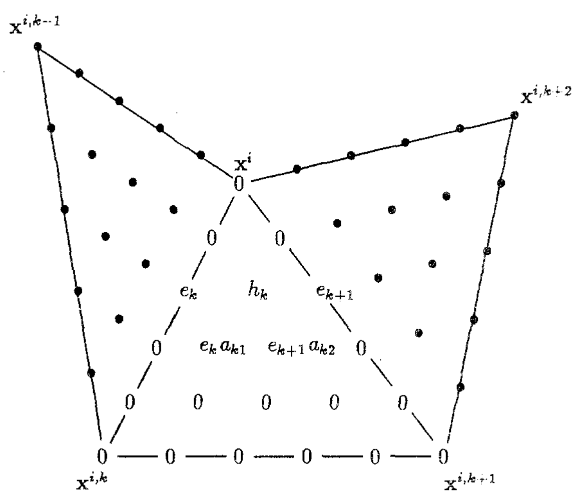

FIG. 7.6. 0-Vertex spline

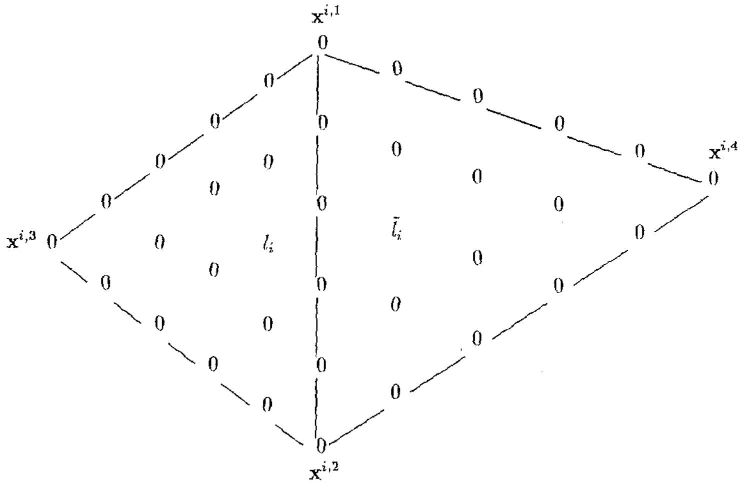

FIG. 7.7. 1-Vertex spline V_{1i}

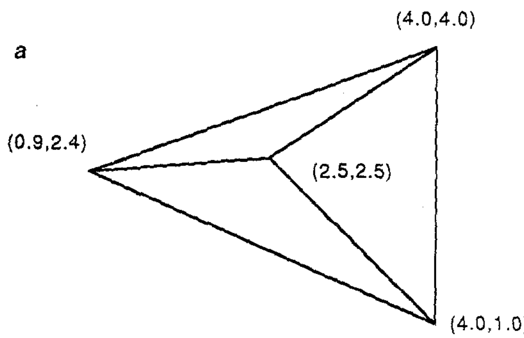

FIG. 7.8a. The support of vertex splines shown in Figs. 7.10–7.15

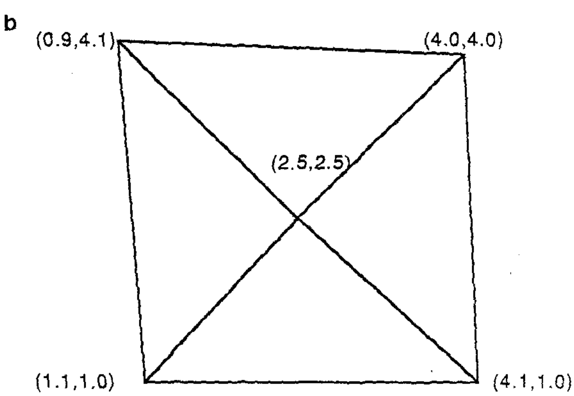

FIG. 7.8b. The support of vertex splines shown in Figs. 7.16–7.21

Suppose that D is a simplicial prtitioned region. Let $\mathbf{x}^i=(x_1^i, x_2^i)$ be a vertex of the partition of D which may be an interior or boundary vertex and $\langle \mathbf{x}^i, \mathbf{x}^{i,k}, \mathbf{x}^{i,k+1}\rangle$, $k=1, ..., l=l(\mathbf{x}^i)$, be the 2-simplices in D which have $\mathbf{x}^i$ as the common vertex, where $\mathbf{x}^{i,l+1}=\mathbf{x}^{i,1}$ if $\mathbf{x}^i$ is an interior vertex. For each $\mathbf{x}^i$, we construct the 0-vertex splines V_{0i}^{γ}, $|\gamma|\leqslant 2$, supported on $\bigcup_{k=1}^{l}\langle \mathbf{x}^i, \mathbf{x}^{i,k}, \mathbf{x}^{i,k+1}\rangle$. For each 1-simplex $\langle \mathbf{x}^{i,1}, \mathbf{x}^{i,2}\rangle$, let $\langle \mathbf{x}^{i,1}, \mathbf{x}^{i,2}, \mathbf{x}^{i,3}\rangle$ and $\langle \mathbf{x}^{i1}, \mathbf{x}^{i,2}, \mathbf{x}^{i,4}\rangle$ be the two 2-simplices whose intersection is $\langle \mathbf{x}^{i,1}, \mathbf{x}^{i,2}\rangle$.

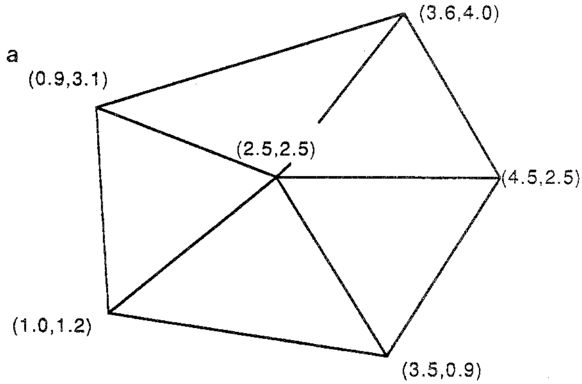

FIG. 7.9a. The support of vertex splines shown in Figs. 7.22–7.27

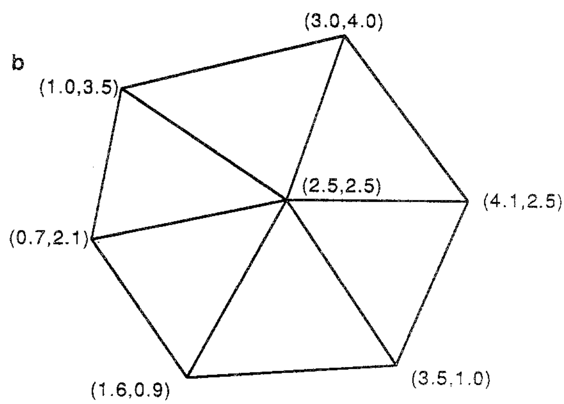

FIG. 7.9b. The support of vertex splines shown in Figs. 7.28–7.33

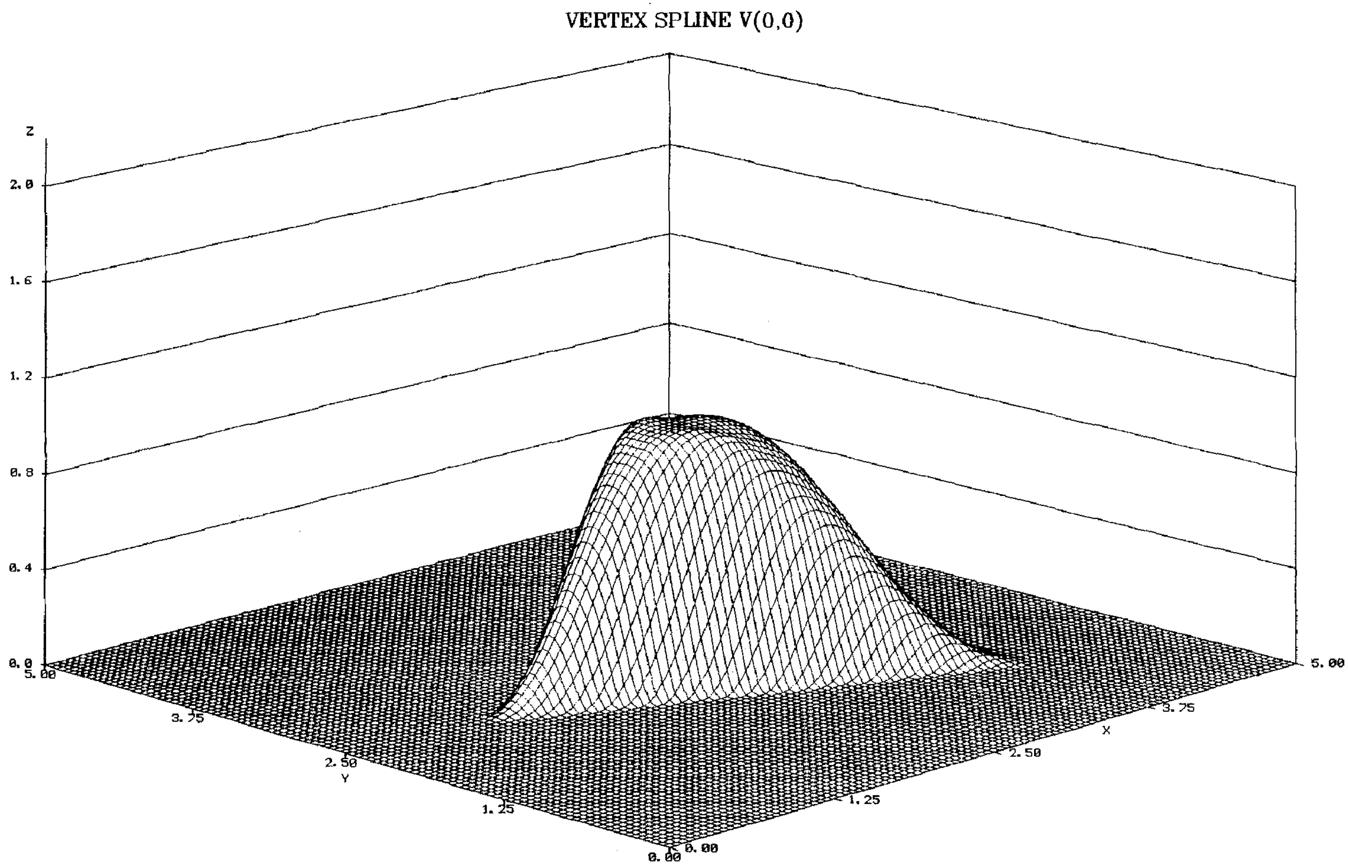

FIG. 7.10. 0-Vertex spline $V_0^{(0,0)}$

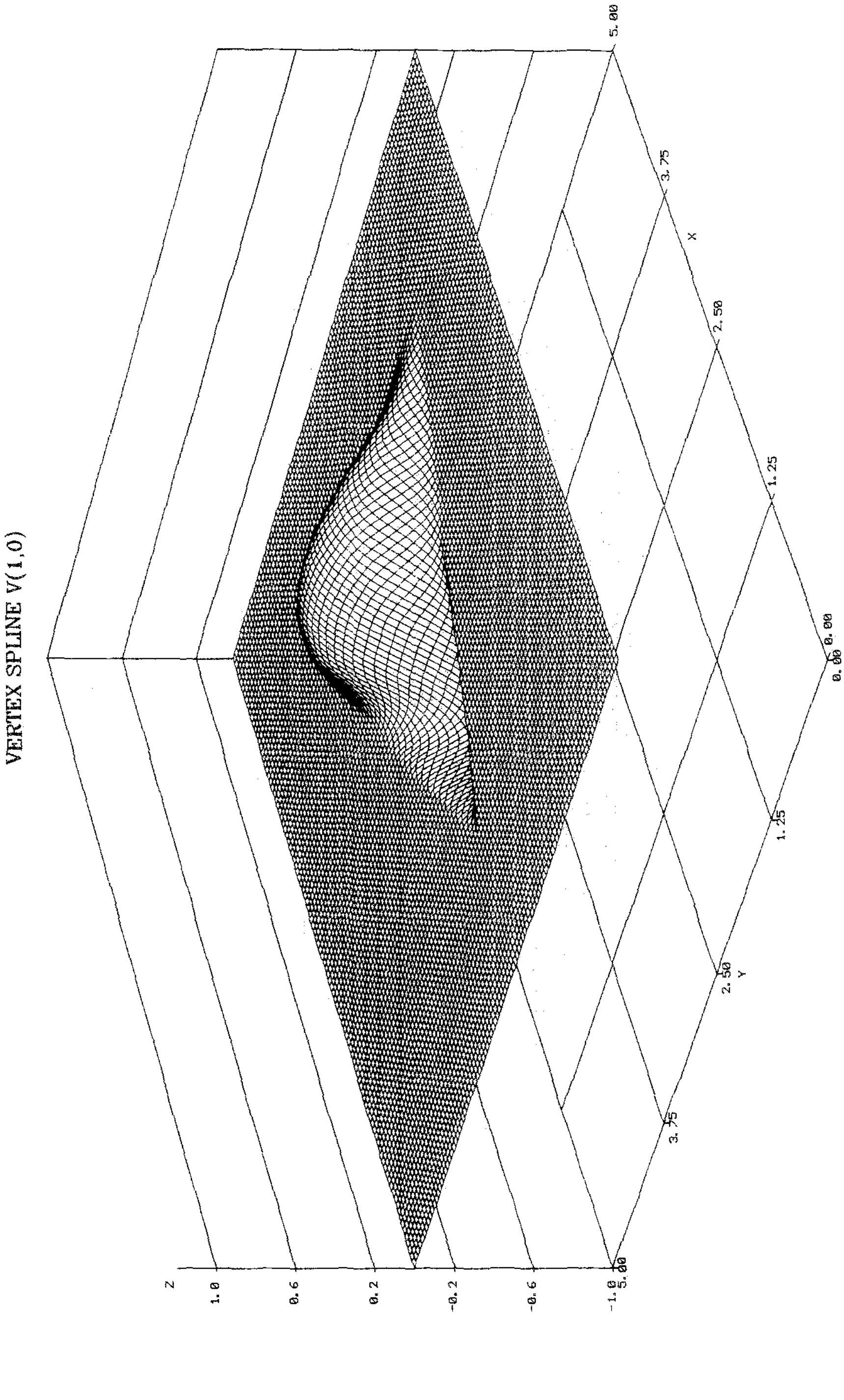

FIG. 7.11. 0-Vertex spline $V_0^{(1,0)}$

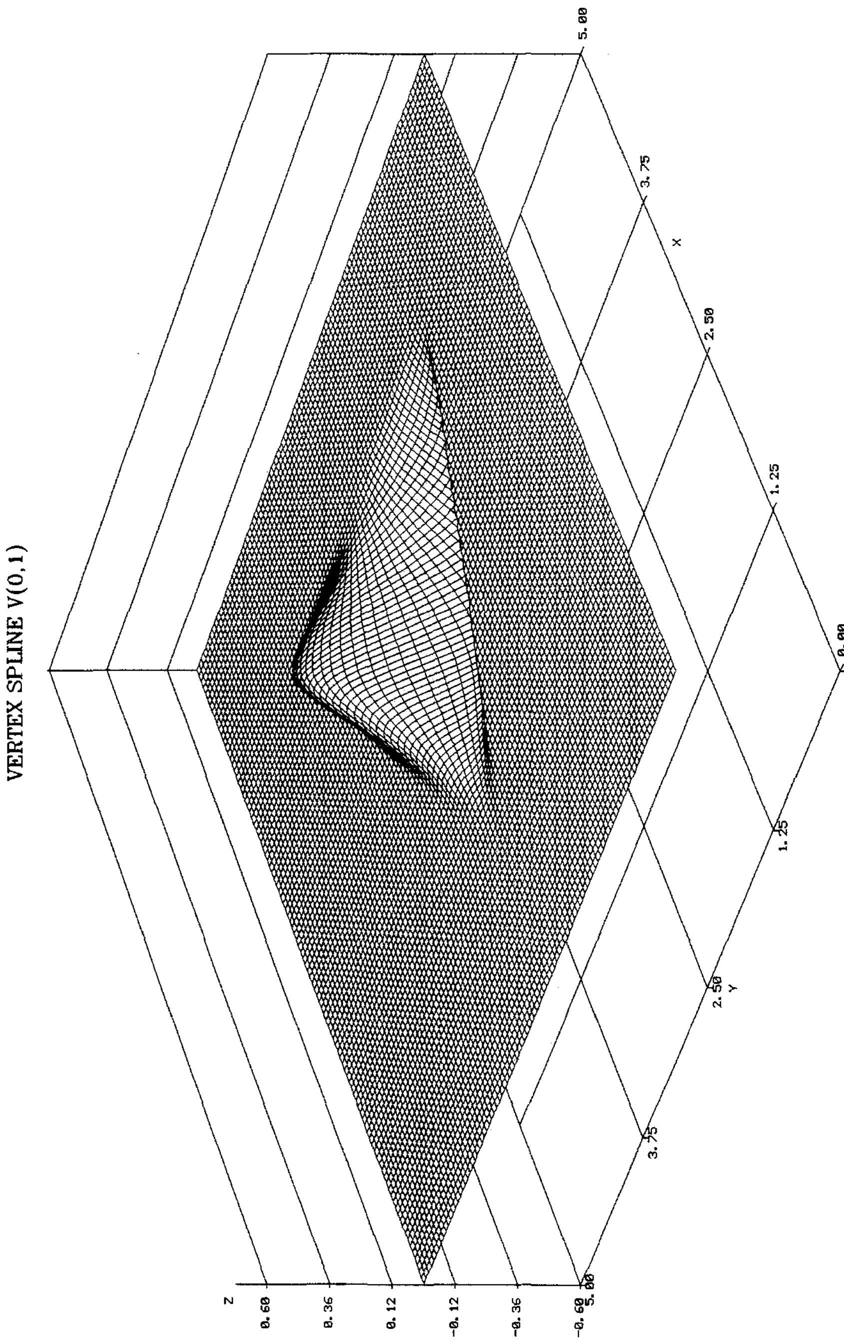

FIG. 7.12. 0-Vertex spline $V_0^{(0,1)}$

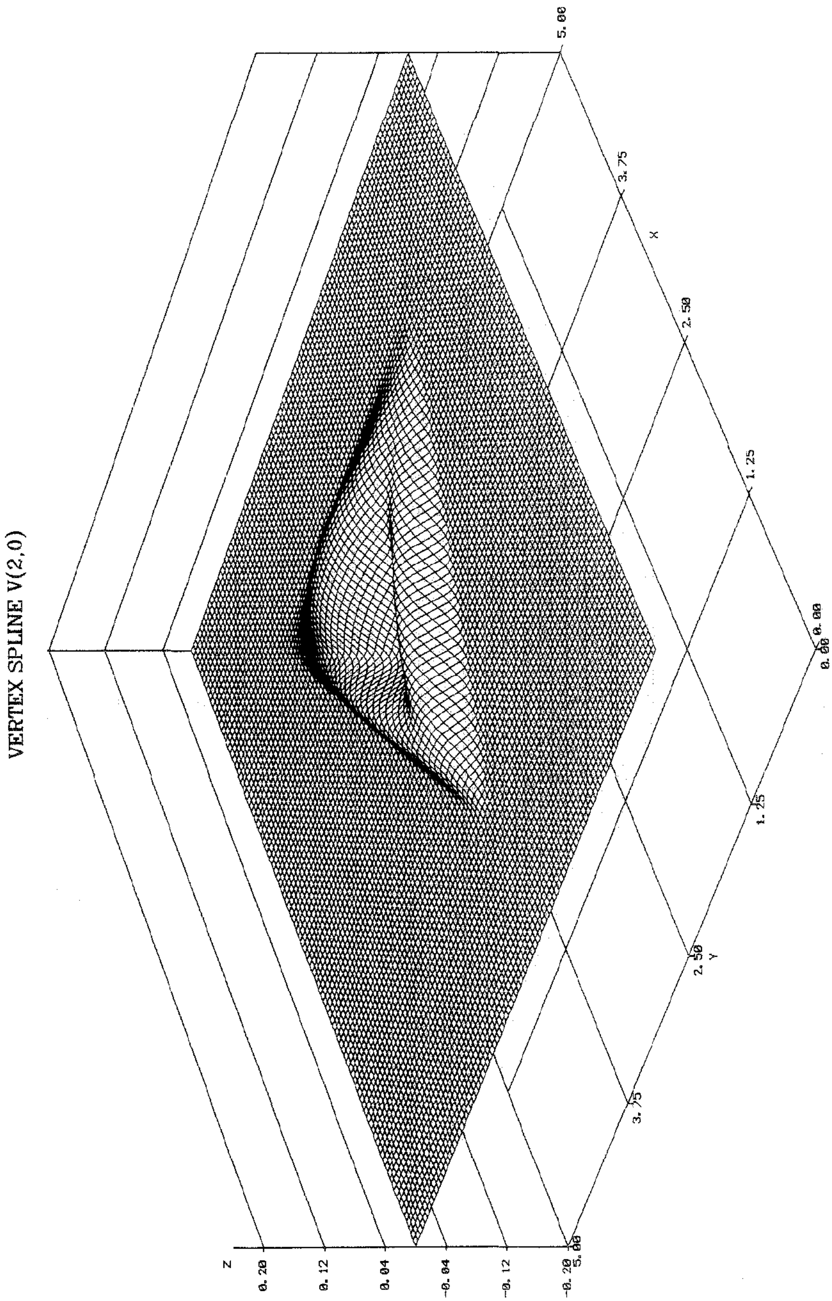

FIG. 7.13. 0-Vertex spline $V_0^{(2,0)}$

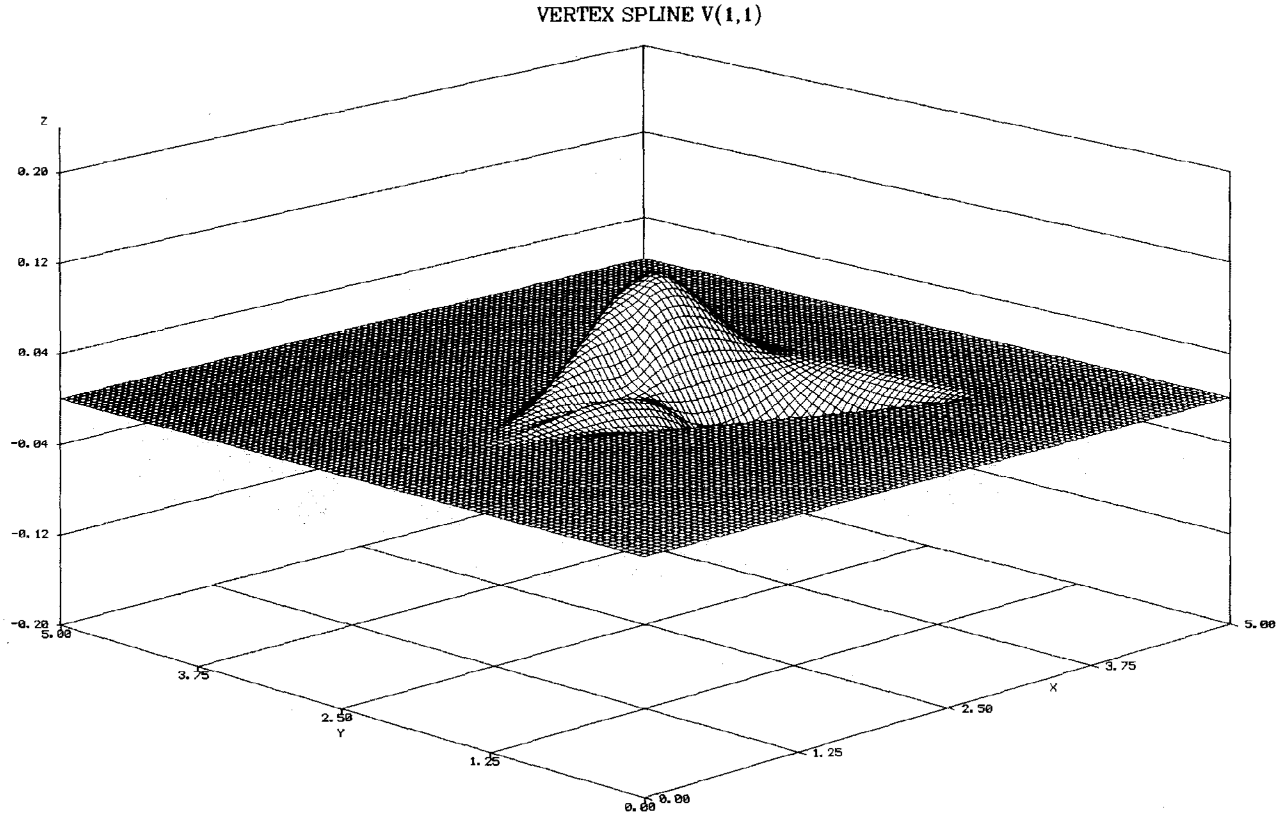

FIG. 7.14. 0-Vertex spline $V_0^{(1,1)}$

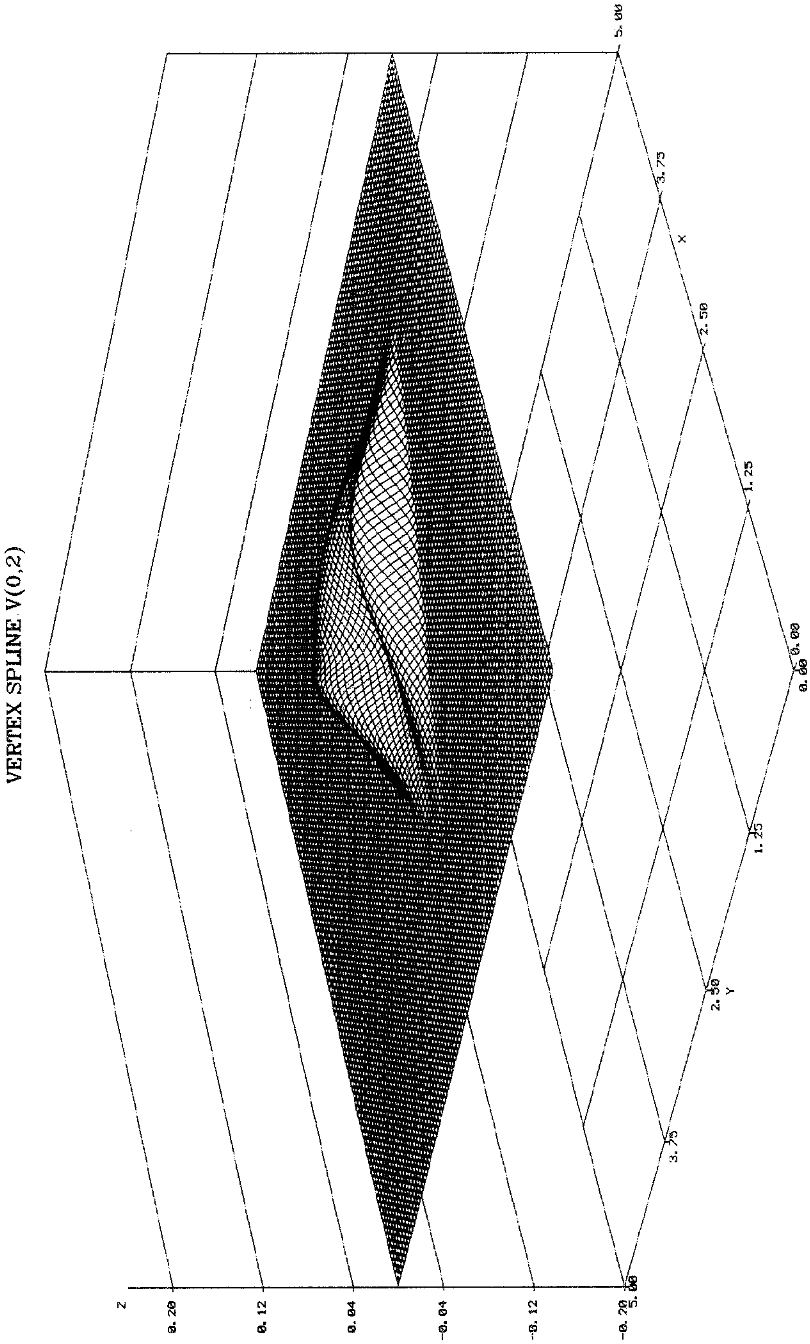

FIG. 7.15. 0-Vertex spline $V_0^{(0,2)}$

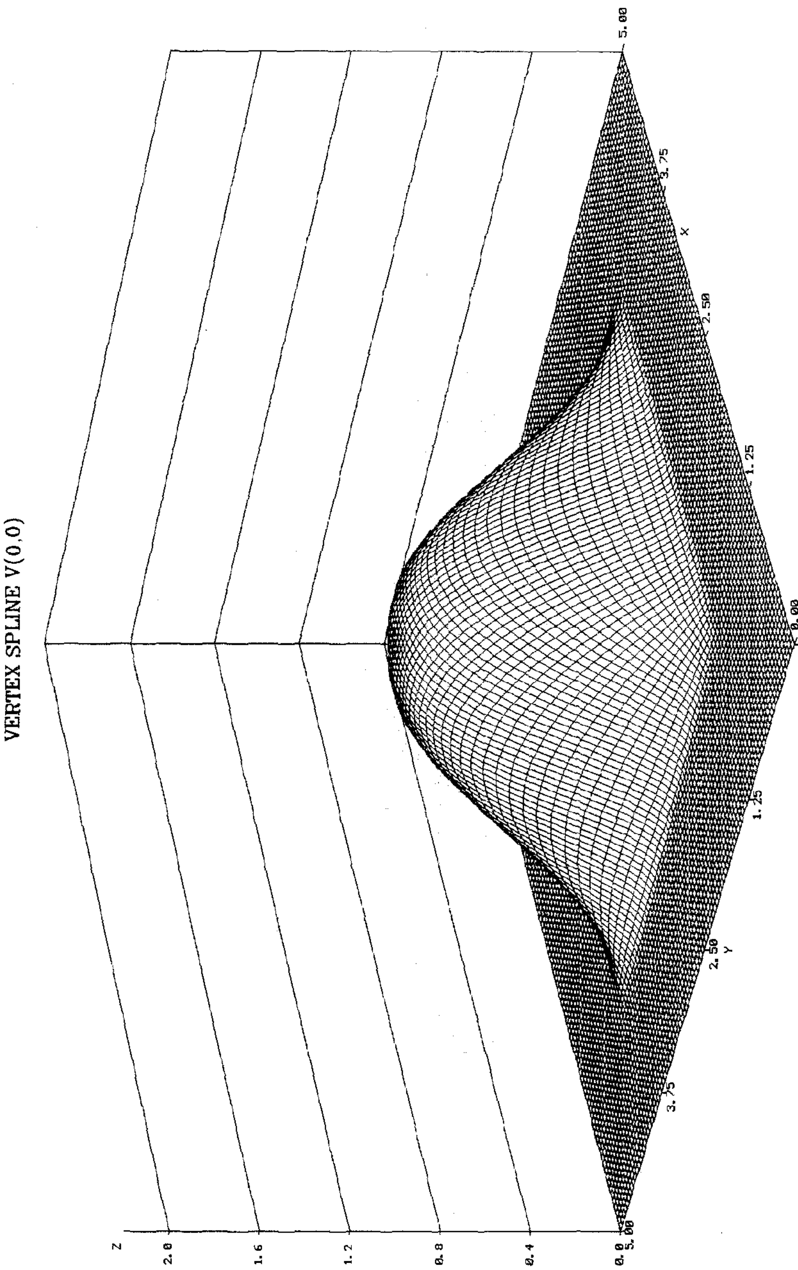

FIG. 7.16. 0-Vertex spline $V_0^{(0,0)}$

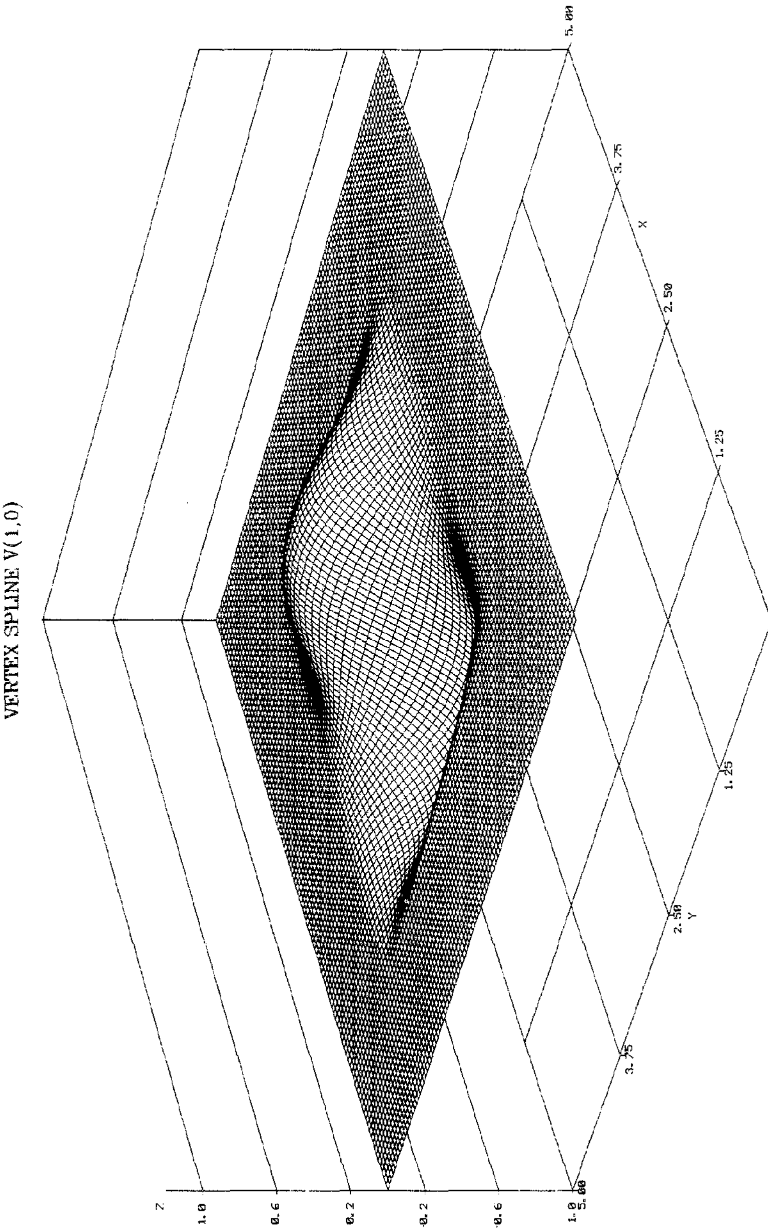

FIG. 7.17. 0-Vertex spline $_0^{(1,0)}$

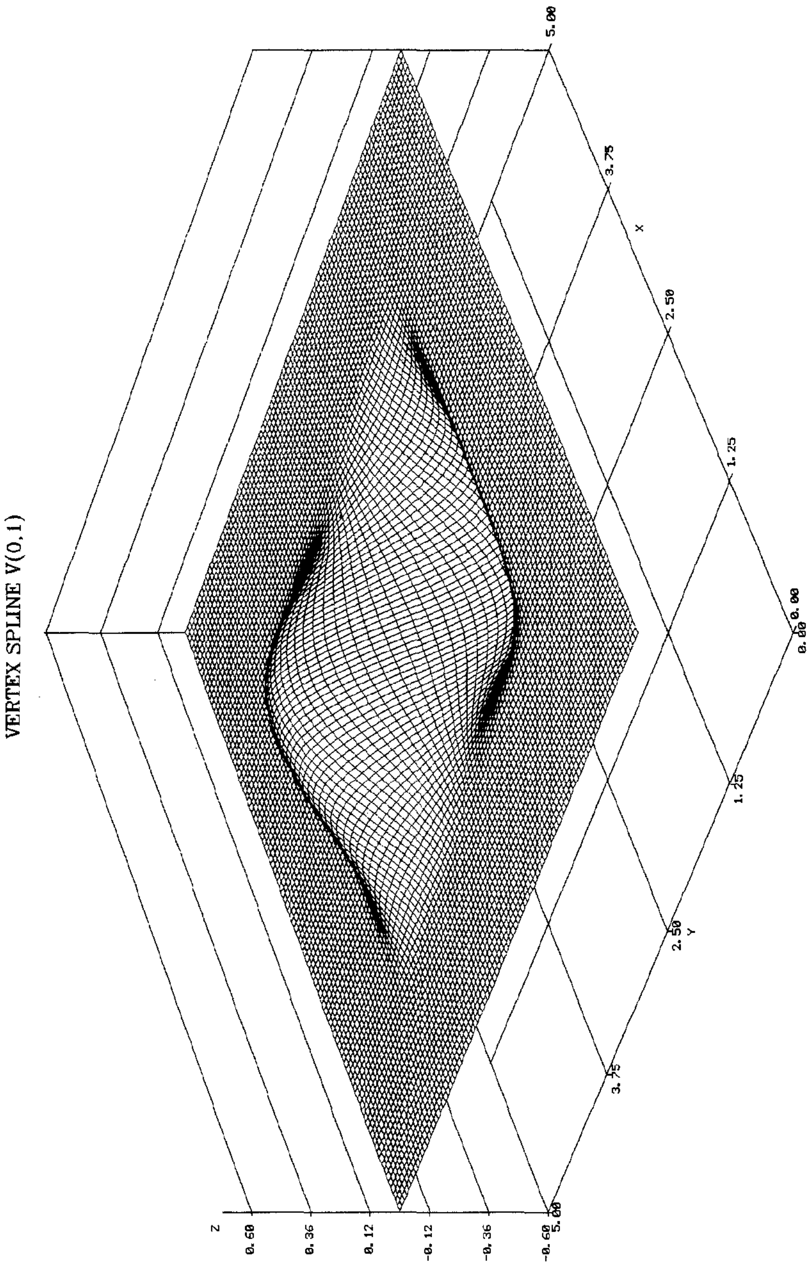

FIG. 7.18. 0-Vertex spline $V_0^{(0,1)}$

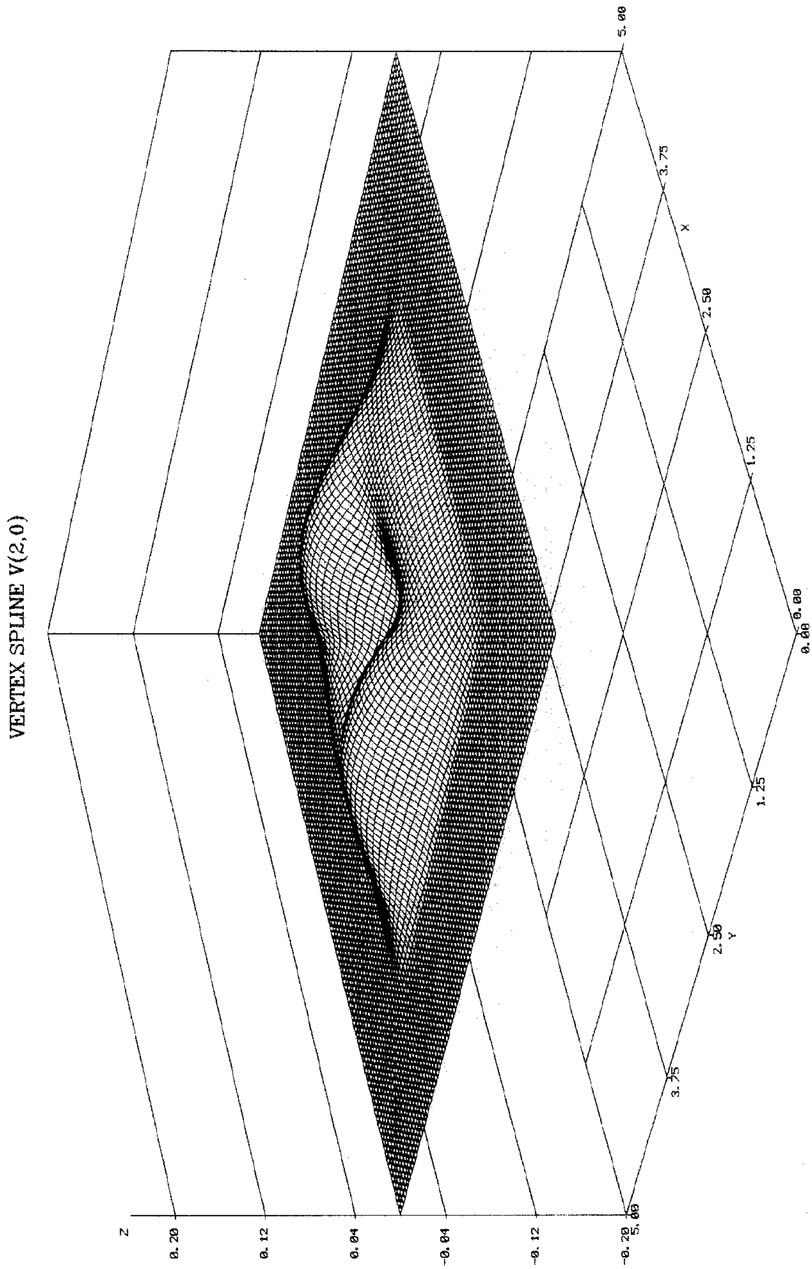

FIG. 7.19. 0-Vertex spline $V_0^{(2,0)}$

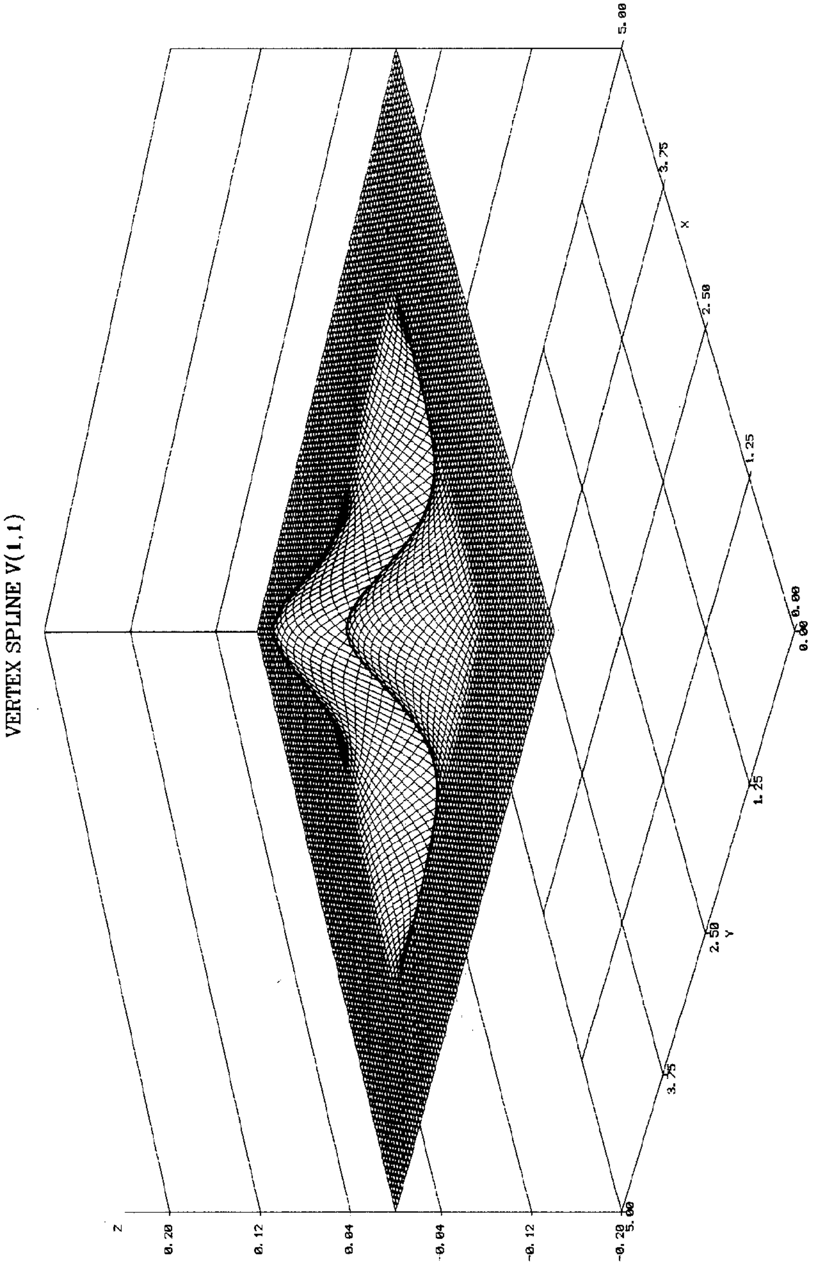

FIG. 7.20. 0-Vertex spline $V_0^{(1,1)}$

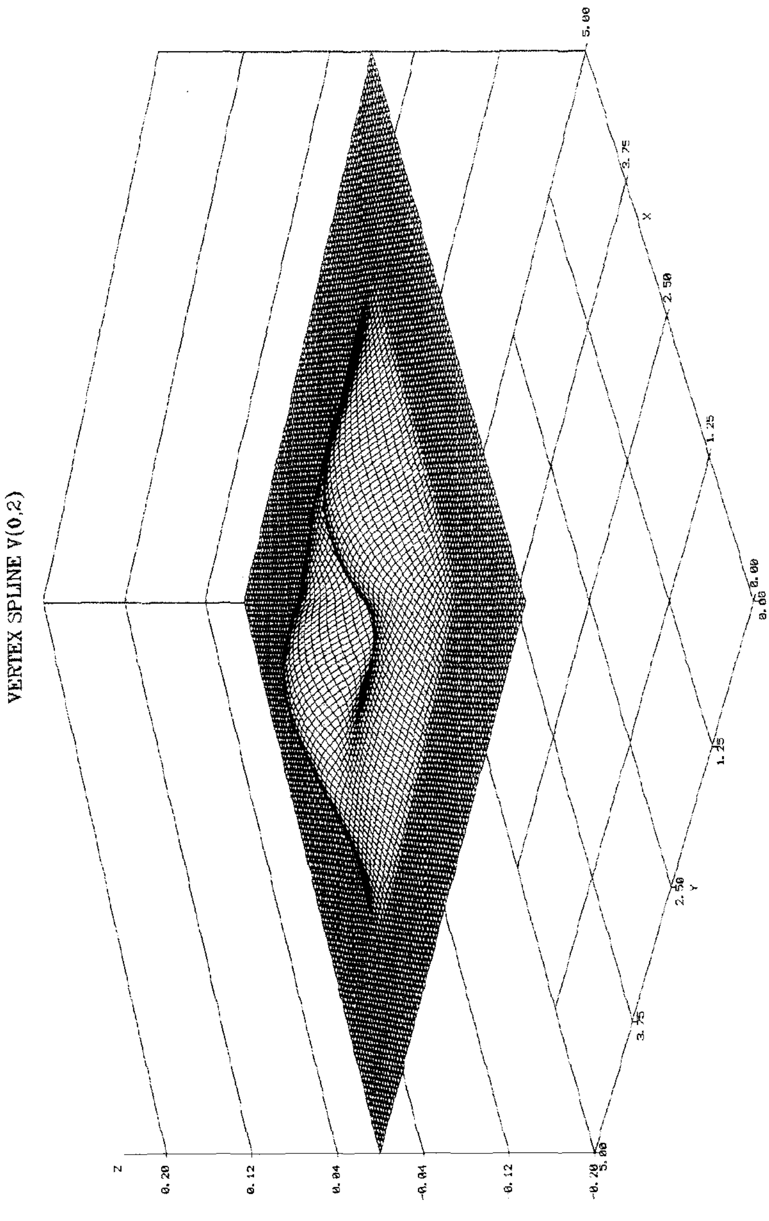

FIG. 7.21. 0-Vertex spline $V_0^{(0,2)}$

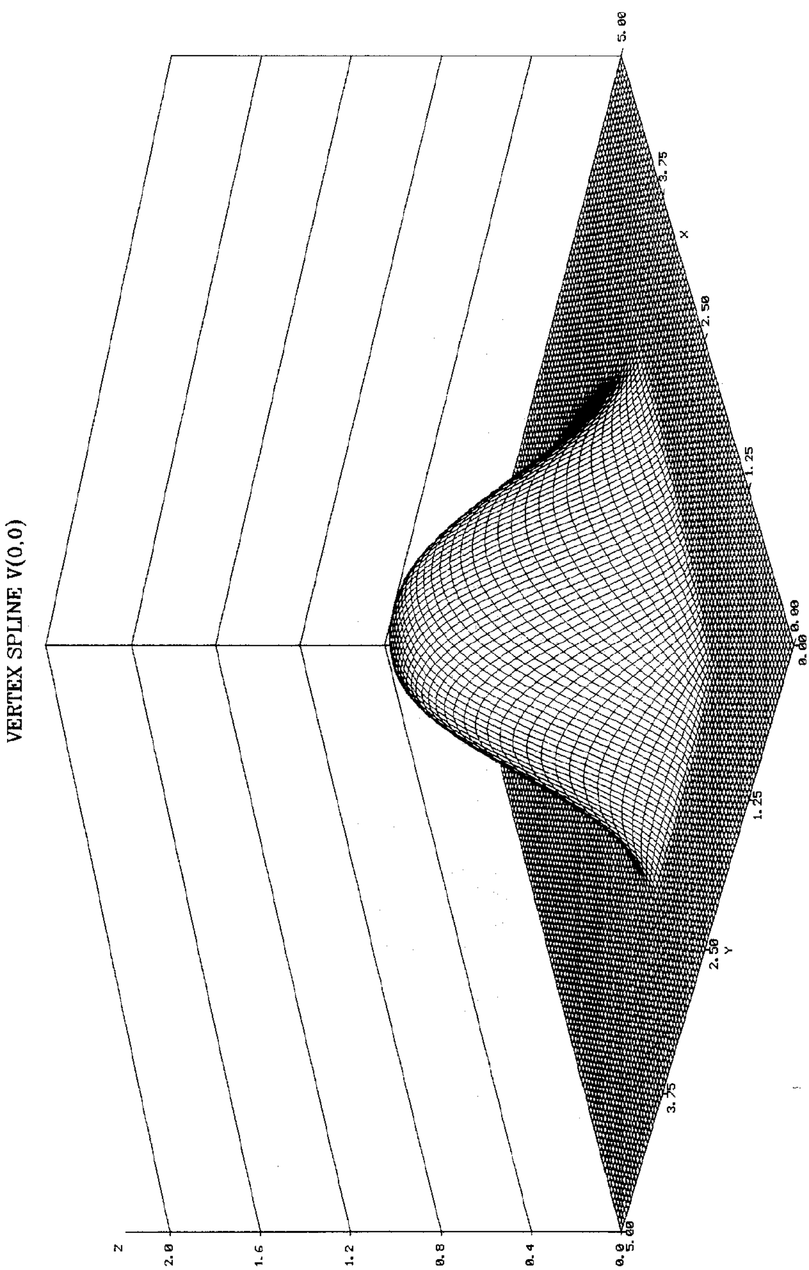

FIG. 7.22. 0-Vertex spline $V_0^{(0,0)}$

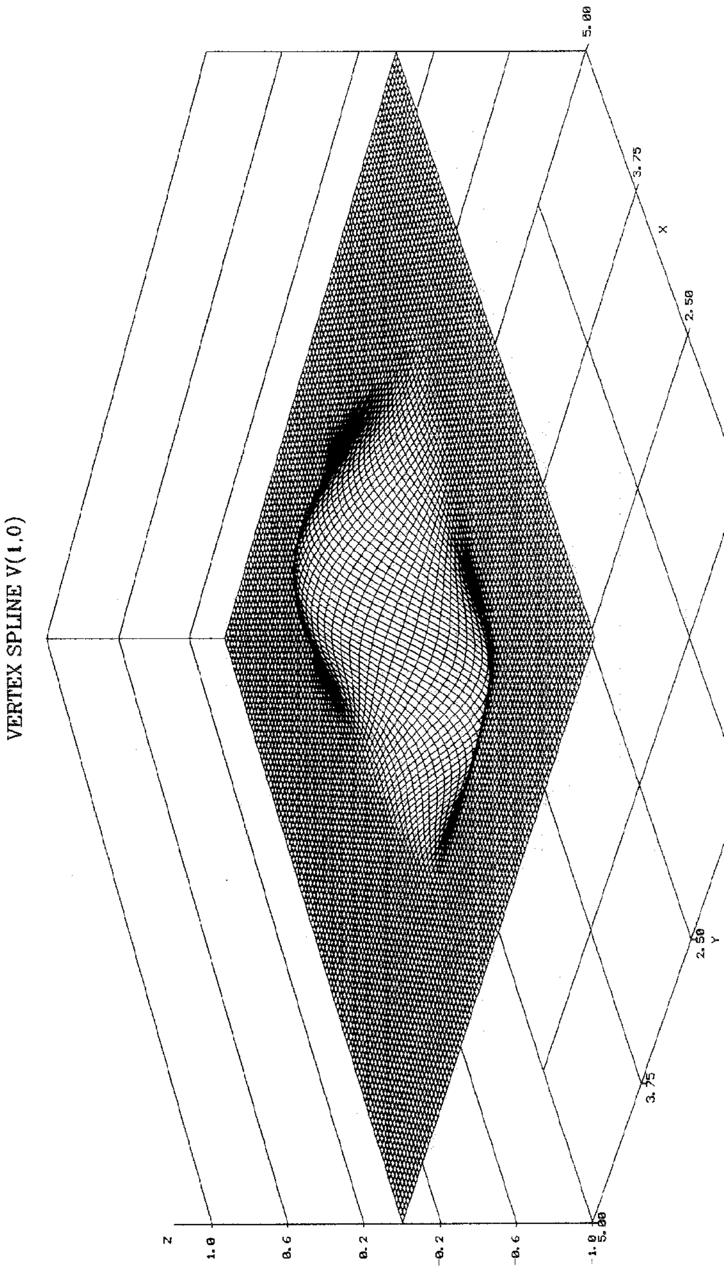

FIG. 7.23. 0-Vertex spline $V_0^{(1,0)}$.

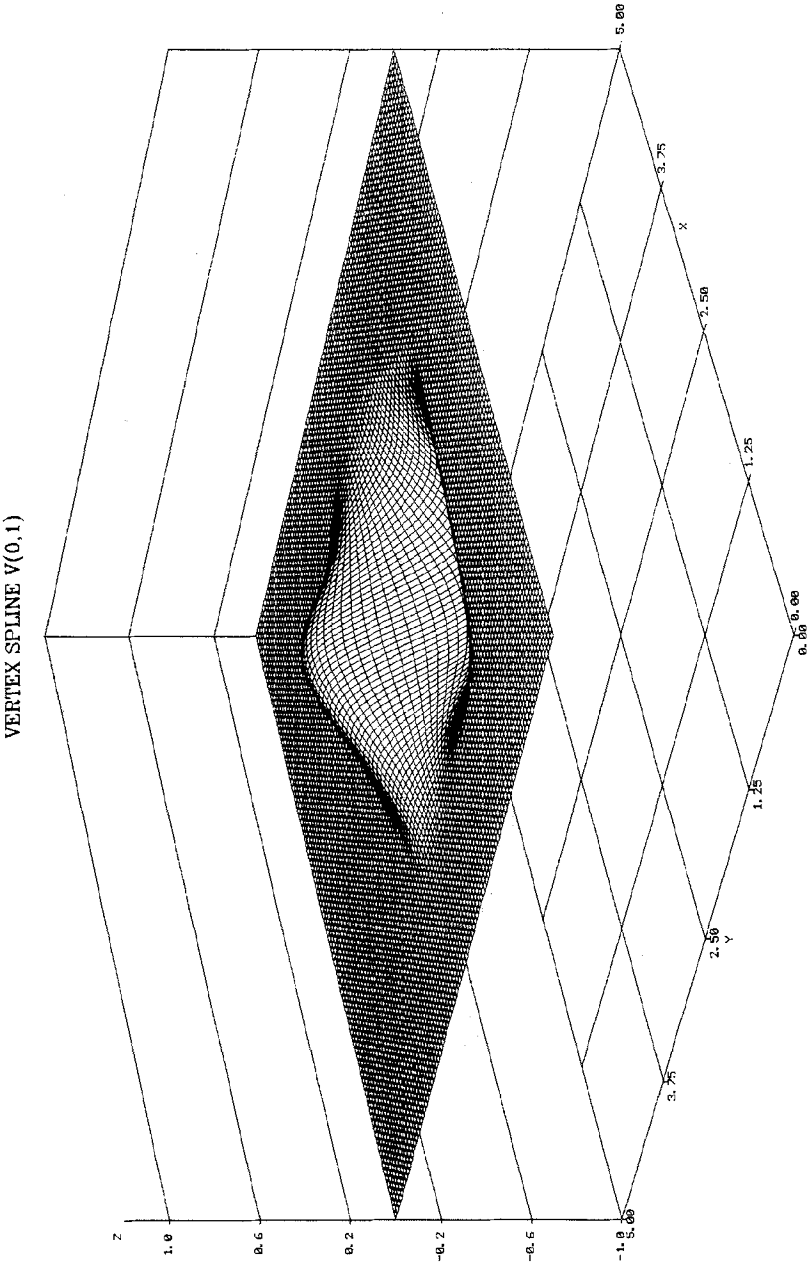

FIG. 7.24. 0-Vertex spline $V_0^{(0,1)}$

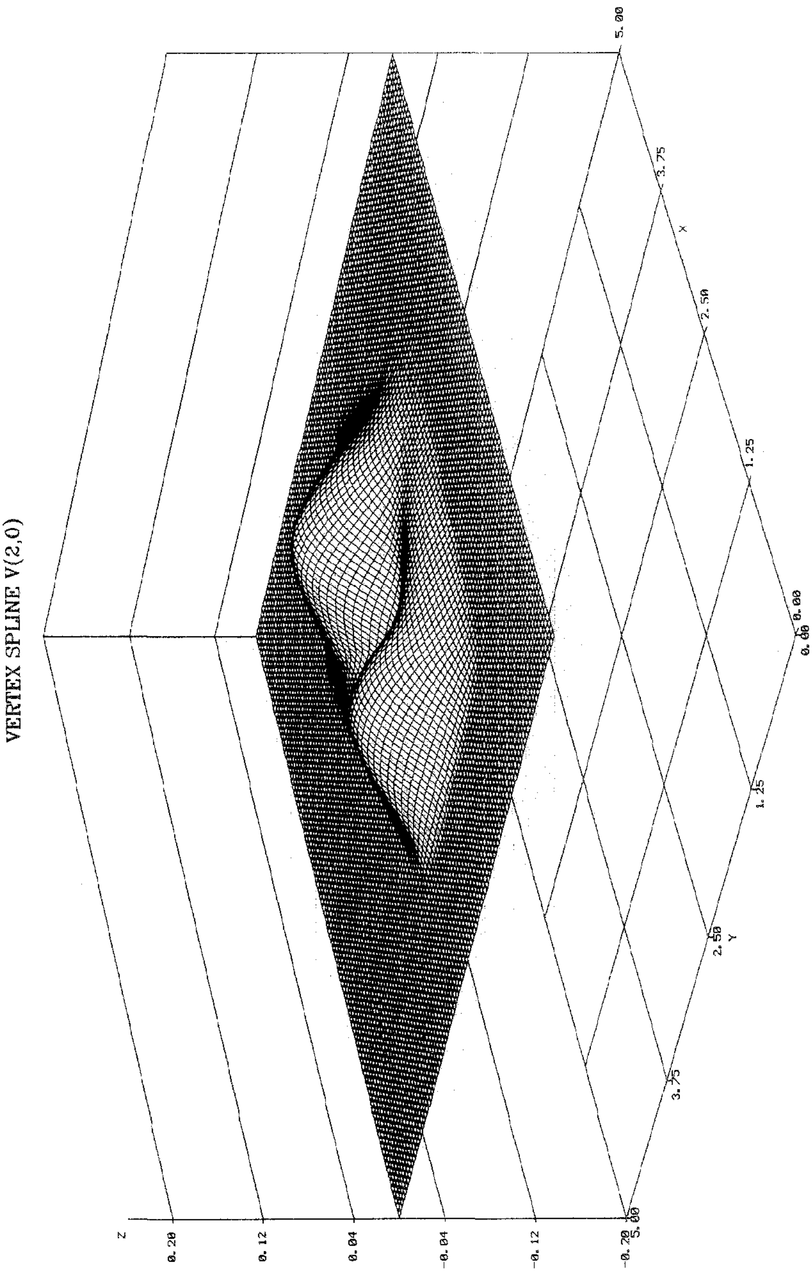

FIG. 7.25. 0-Vertex spline $V_0^{(2,0)}$

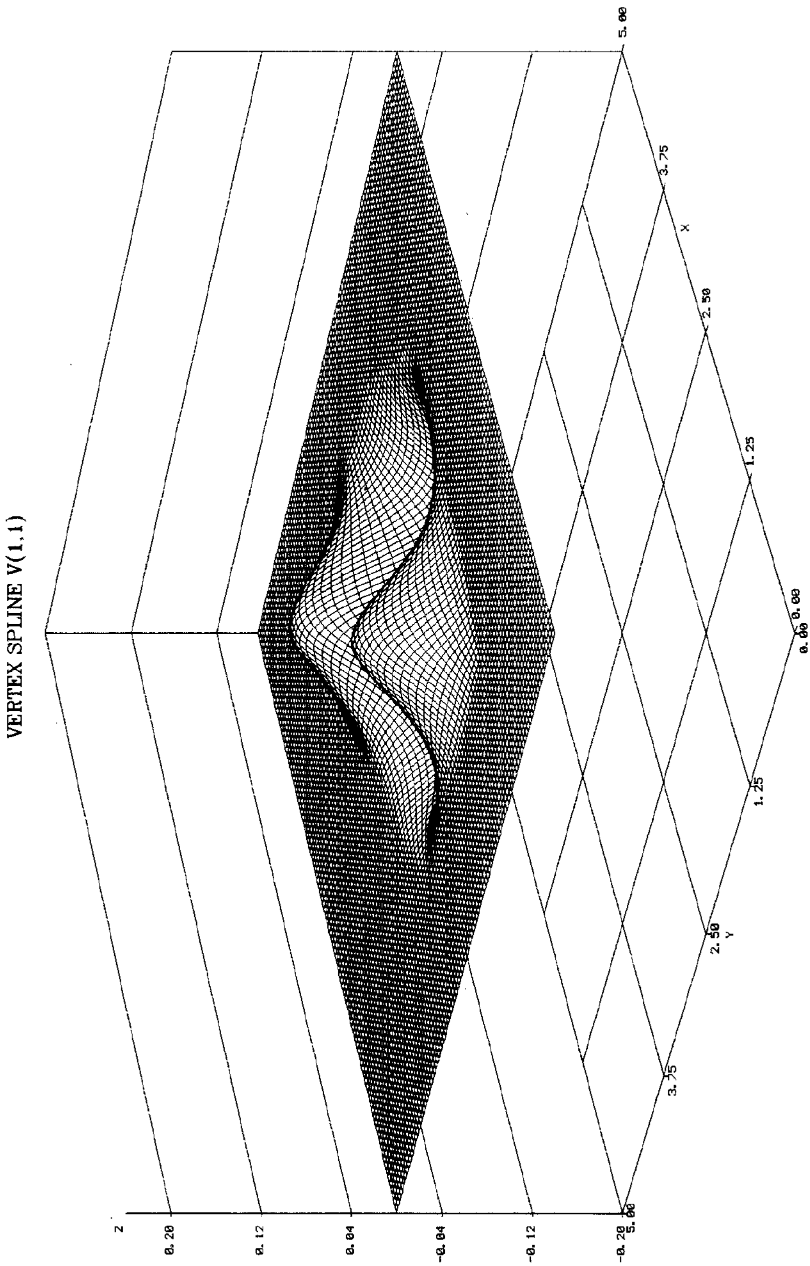

FIG. 7.26. 0-Vertex spline $V_0^{(1,1)}$

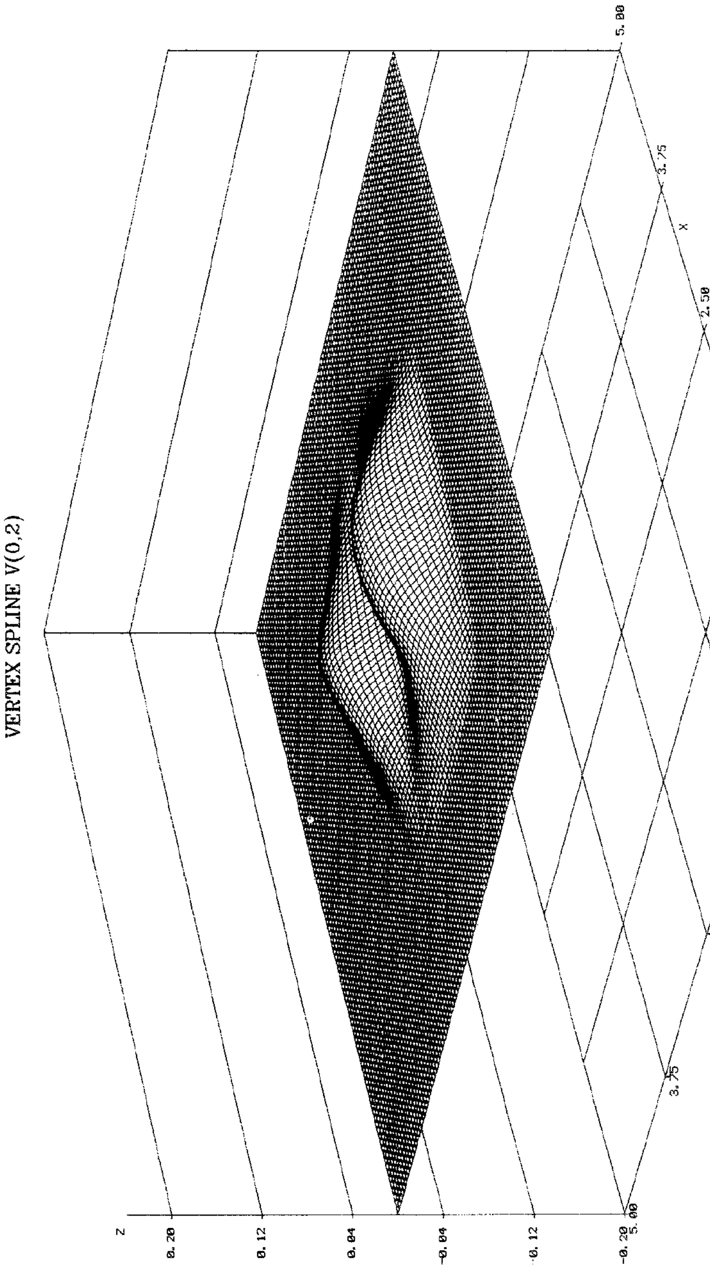

FIG. 7.27. 0-Vertex spline $V_0^{(0,2)}$

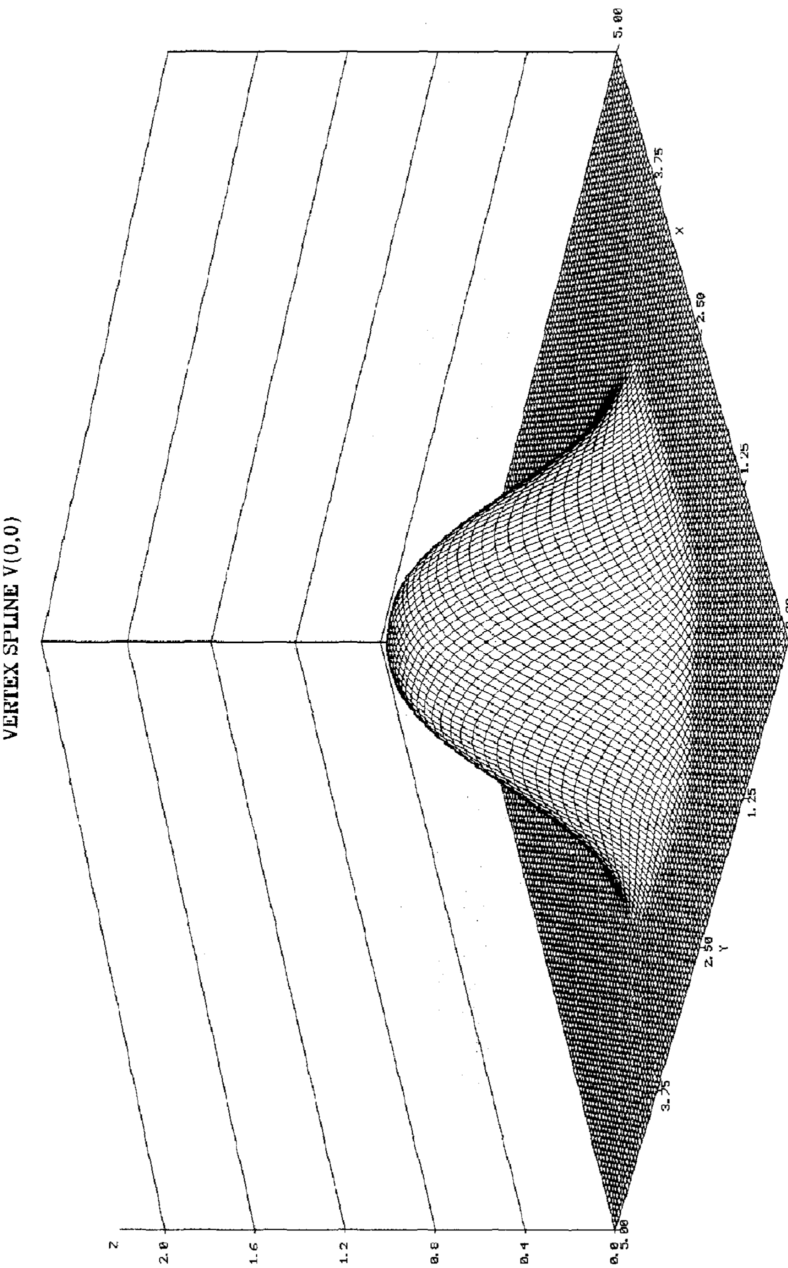

FIG. 7.28. 0-Vertex spline $V_0^{(0,1)}$

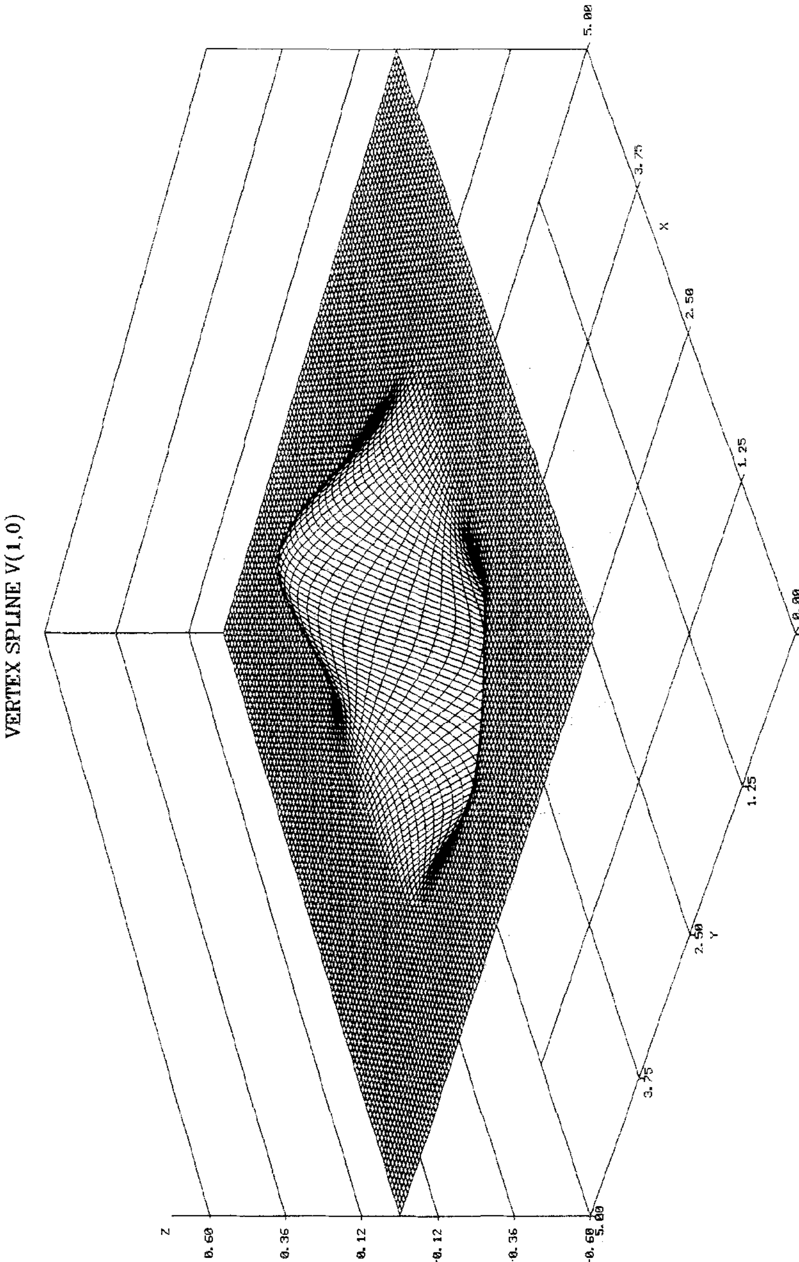

FIG. 7.29. 0-Vertex spline $V_0^{(1,0)}$

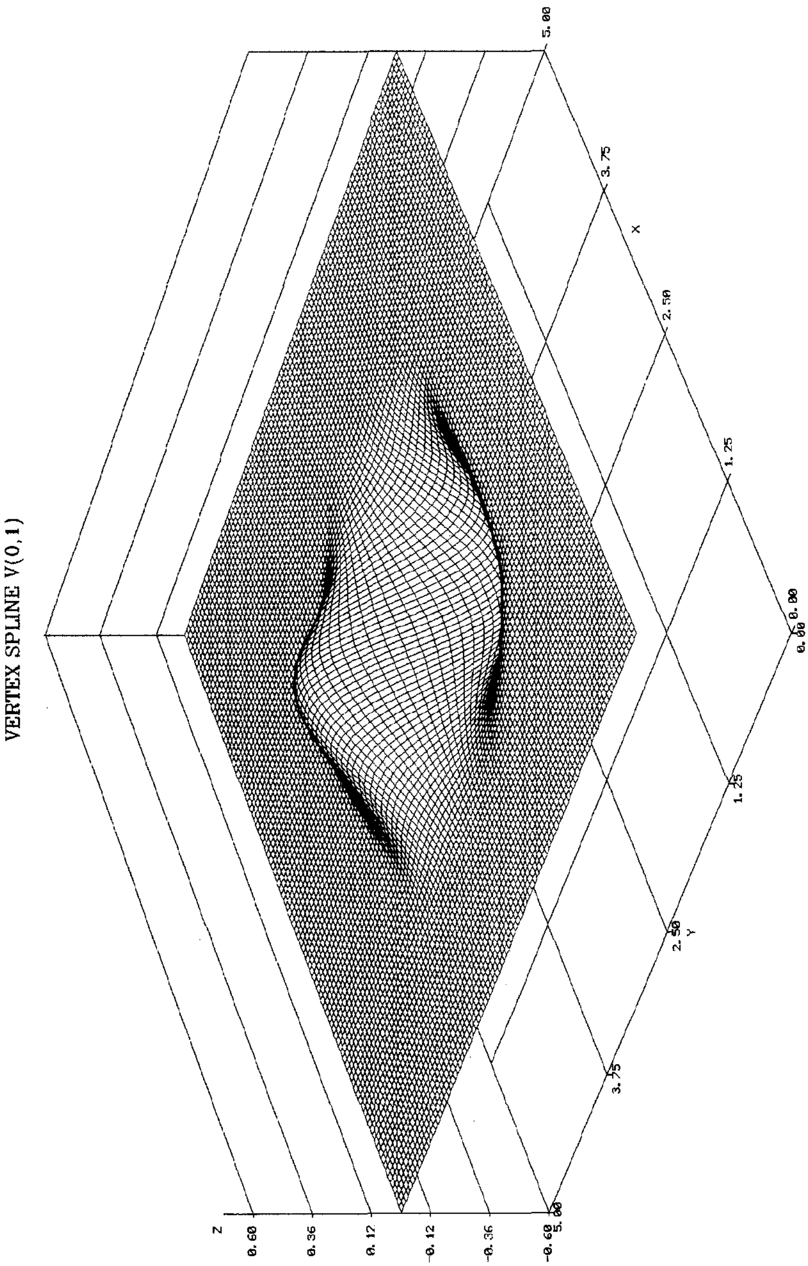

FIG. 7.30. 0-Vertex spline $V_0^{(0,1)}$

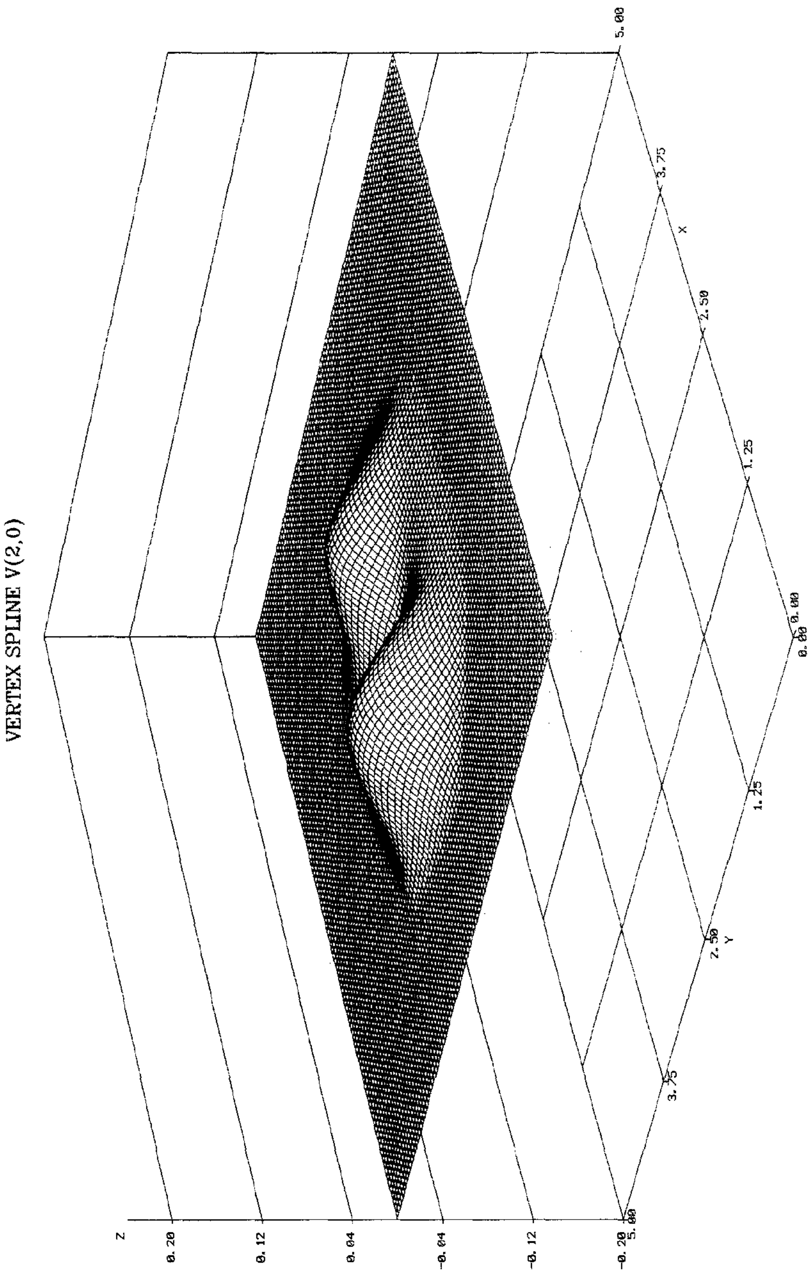

FIG. 7.31. 0-Vertex spline $V_0^{(2,0)}$

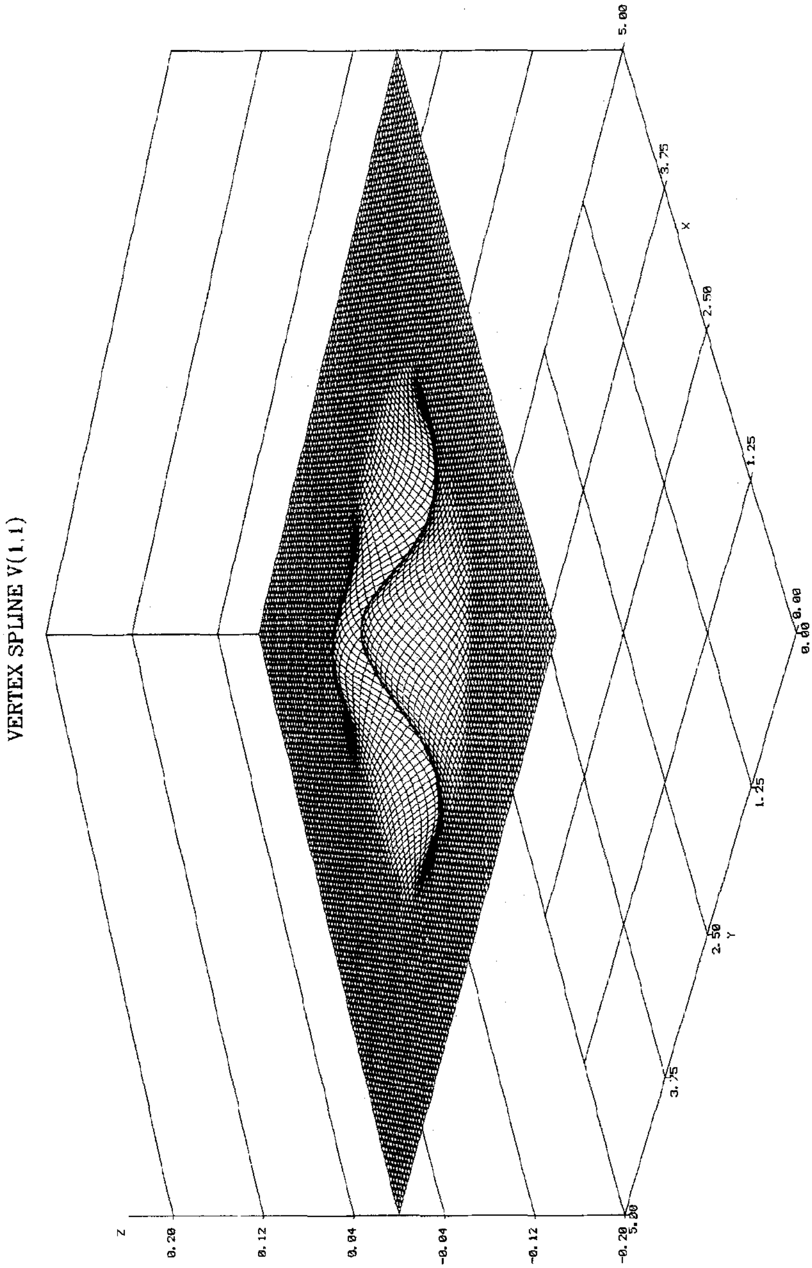

FIG. 7.32. 0-Vertex spline $V_0^{(1,1)}$

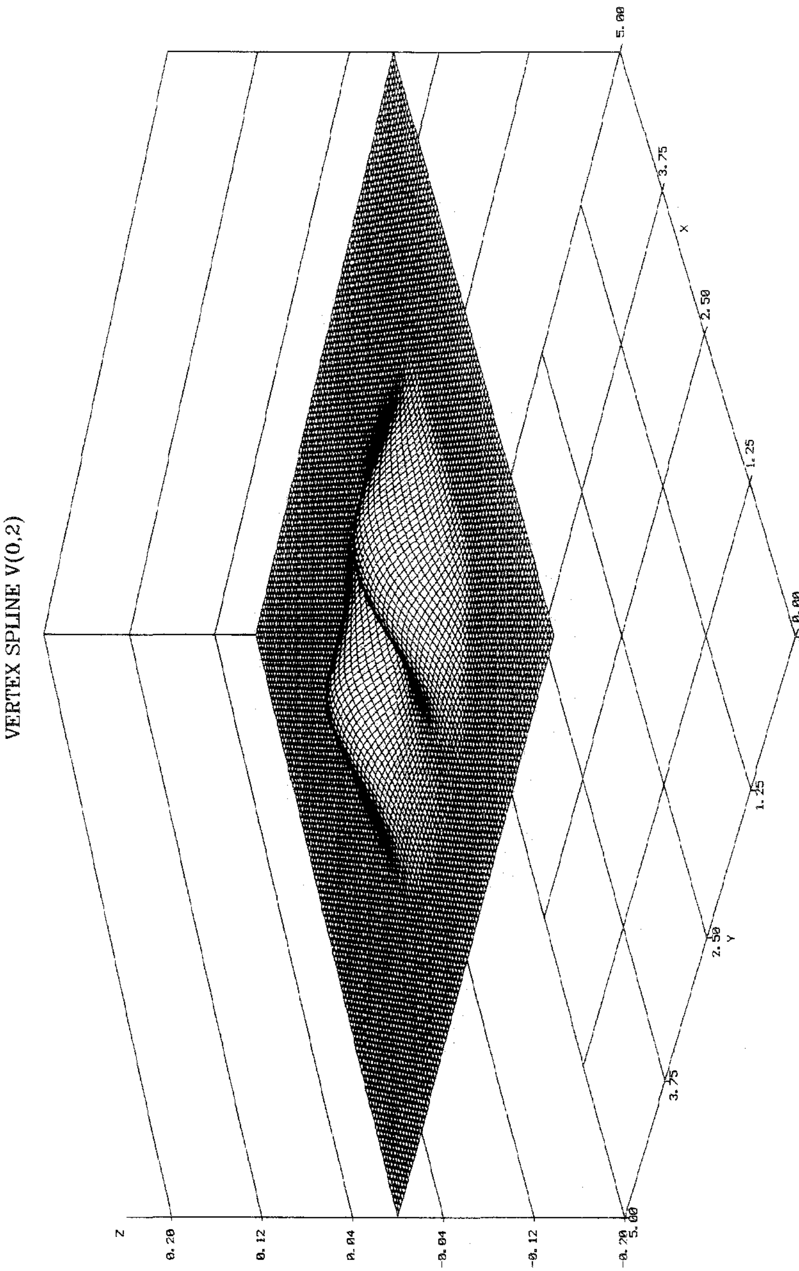

FIG. 7.33. 0-Vertex spline $V_0^{(0,2)}$

We construct the 1-vertex spline V^{γ}_{1i} supported on the union of these two 2-simplices. The Bézier nets of these vertex splines are displayed in Figs. 7.1–7.7. Set

$$a_{k1}=\frac{\delta(\mathbf{x}^{i,k},\mathbf{x}^{i,k+1},\mathbf{x}^{i,k-1})}{\delta(\mathbf{x}^{i,k+1},\mathbf{x}^{i},\mathbf{x}^{i,k-1})+\delta(\mathbf{x}^{i,k},\mathbf{x}^{i,k+1},\mathbf{x}^{i,k-1})},$$

$$a_{k2}=\frac{\delta(\mathbf{x}^{i,k+1},\mathbf{x}^{i,k+2},\mathbf{x}^{i,k})}{\delta(\mathbf{x}^{i,k+2},\mathbf{x}^{i},\mathbf{x}^{i,k})+\delta(\mathbf{x}^{i,k+1},\mathbf{x}^{i,k+2},\mathbf{x}^{i,k})},$$

$$b_k=\frac{1}{5}(x_1^{i,k}-x_1^i),\qquad c_k=\frac{1}{5}(x_2^{i,k}-x_2^i),$$

$$d_k=\frac{1}{20}(x_1^{i,k}-x_1^i)^2,$$

$$e_k=\frac{1}{20}(x_2^{i,k}-x_2^i)^2,$$

$$f_k=\frac{1}{20}(x_1^{i,k}-x_1^i)(x_1^{i,k+1}-x_1^i),$$

$$g_k=\frac{1}{10}(x_1^{i,k}-x_1^i)(x_2^{i,k}-x_2^i),$$

$$\tilde{g}=\frac{1}{20}[(x_1^{i,k+1}-x_1^i)(x_2^{i,k}-x_2^i)+(x_2^{i,k+1}-x_2^i)(x_1^{i,k}-x_1^i)],$$

$$h_k=\frac{1}{20}(x_2^{i,k}-x_2^i)(x_2^{i,k+1}-x_2^i),$$

and

$$l_i=\delta(\mathbf{x}^{i,1},\mathbf{x}^{i,2},\mathbf{x}^{i,3})\qquad \bar{l}_i=\delta(\mathbf{x}^{i,1},\mathbf{x}^{i,2},\mathbf{x}^{i,4}),$$

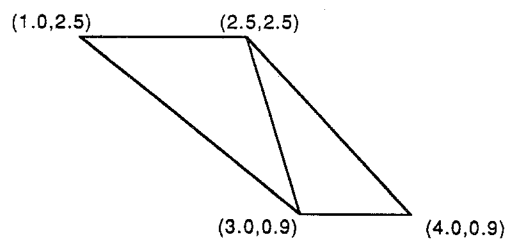

FIG. 7.34. The support of the 1-vertex spline shown in Fig. 7.35

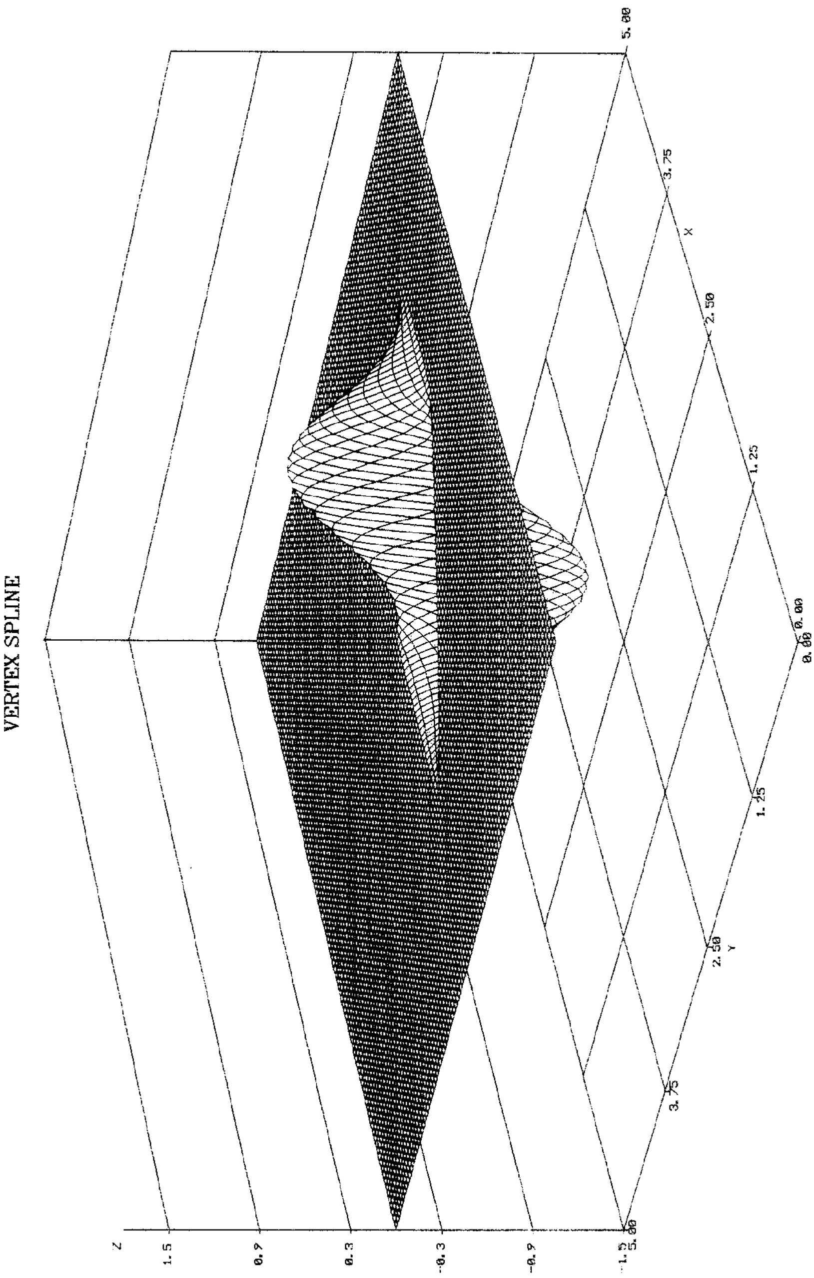

FIG. 7.35. 1-Vertex spline V_1.

where, as usual, we write $\mathbf{x}^i = (x_1^i, x_2^i)$, $\mathbf{x}^{i,k} = (x_1^{i,k}, x_2^{i,k})$ and denote by

$$\delta(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3) = \frac{1}{2} \begin{vmatrix} 1 & x_1^1 & x_2^1 \\ 1 & x_1^2 & x_2^2 \\ 1 & x_1^3 & x_2^3 \end{vmatrix}$$

the signed area of the 2-simplex $\langle \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \rangle$.

We conclude with graphs of vertex splines on various supports (Figs. 7.8–7.35).

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